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We have used a novel computer with a novel integration technique to study the evolution of the entire planetary system for nearly 100 million years. This calculation confirms that the evolution of the Solar System as a whole is chaotic, with a remarkably short timescale of exponential divergence of about 4 million years. Additional numerical experiments indicate that the dynamical evolution of the Jovian planets is chaotic apart from the rest of the system, as is the motion of Pluto.

Advances in computer technology have made it possible to begin to directly address the age-old question of the nature of the long-term evolution of the Solar System, with startling results. Sussman and Wisdom [1] presented numerical evidence that the motion of Pluto is chaotic, with a timescale for exponential divergence of nearby trajectories of only about 20 million years. Subsequently, Laskar [2] found numerical evidence of the chaotic evolution of the whole Solar System excluding Pluto, with a timescale for exponential divergence of only about 5 million years. Laskar's calculation was feasible because he analytically averaged the equations of motion to remove the rapid variations with timescales of order the orbital period. The averaged equations are perturbative and necessarily truncated after a particular order in eccentricity, inclination and mass ratio. A new integration of the whole Solar System without these approximations was clearly required.

Direct integrations of the whole planetary system are computationally expensive. Notable long-term integrations of the outer Solar System include: the classic 1 million year integration of Cohen, Hubbard, and Oesterwinter [3], the 5 million year integration of Kinoshita and Nakai [4], the 210 million year integration performed on the Digital Orrery [5], the 100 million year integration of the LONGSTOP project [6], and the 845 million year Digital Orrery integration of Sussman and Wisdom [1]. Studies of the long-term evolution of the whole Solar System have been more limited because the computational resources required are significantly larger, by about two orders of magnitude. Integrations of the whole Solar System include: the 3 million year

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Digital Orrery integration (excluding Mercury) [5], the 2 million year integration of Richardson and Walker [7], and the recent ± 3 million year integration of Quinn, Tremaine, and Duncan [8, 9] (hereafter QTD).

We have developed new computational techniques and new computer hardware to make possible a direct integration of the whole Solar System spanning a significantly longer interval than previously achieved. Our new direct integration of the equations of motion spans 36,000,000,000 days, or about 98.6 million years. Our earlier result concerning the chaotic motion of Pluto, as well as the result of Laskar that the Solar System is chaotic are both confirmed. In order to localize the sources of the chaotic behavior we have carried out numerous additional long-term integrations. We have found that the evolution of the Jovian planets is independently chaotic, as is the motion of Pluto.

Method of Integration

We use the symplectic n-body mapping method of Wisdom and Holman [10] to integrate the planetary system. This new mapping method is a generalization of the mapping method of Wisdom [11, 12]. The basic idea is as follows.

The Hamiltonian for the planetary n-body problem can be written

$$H = H_{Kepler} + H_{Interaction}, \tag{1}$$

where the first term represents the Keplerian motion of each of the planets with respect to the sun, and the second term describes the planetary perturbations. The simplest form of the map is obtained by adding short period terms so that the Hamiltonian becomes

$$H_{Map} = H_{Kepler} + H_{Interaction} 2\pi\delta_{2\pi}(\Omega t), \tag{2}$$

where $\delta_{2\pi}(t)$ is a periodic sequence of Dirac delta functions with period 2π, and Ω is the mapping frequency. The averaging principle [13] suggests that the additional short period terms do not significantly affect the long-period evolution. This Hamiltonian is locally integrable: between the delta functions the motion is purely Keplerian and integration across the delta functions is easily carried out in Cartesian coordinates. The resulting map of the phase space onto itself is a composition of canonical transformations and is thus canonical (i.e., the Jacobian is a symplectic matrix). In general, to make a mapping of this sort each part of the Hamiltonian, considered by itself, must be not only integrable but also efficiently solvable. The Keplerian phase of the evolution can be efficiently carried out directly in Cartesian coordinates with the aid of Gauss's f and g functions. The uniform use of the Cartesian coordinates also eliminates any need for costly intermediate coordinate transformations.

The mapping method can be made accurate to arbitrarily high-order in the mapping step-size. This and other refinements are discussed in Wisdom and Holman [10]. In the applications presented here the second order version of the n-body mapping is used. Second order is achieved by evolving the system with the Kepler Hamiltonian for a half mapping step, followed by an alternating

succession of full interaction kicks and whole Keplerian steps, but ending with a half Kepler step.

This ridiculously simple scheme is actually a remarkably good and efficient integrator. Wisdom and Holman [10] used this new map to compute the evolution of the outer planets for a billion years and compared the results to the results of the 845 million year integrations performed on the Digital Orrery by Sussman and Wisdom [1] using conventional integration techniques. All of the results of that study were confirmed, including details of the very long-period variations (> 500 million years) in the inclination of Pluto, and the 20 million year exponential divergence of trajectories, which indicated chaotic behavior in the motion of Pluto. The new map is about an order of magnitude faster than traditional methods of integration.

Our 100 Million Year Integration

Our 100 million year integration of the planetary system was performed using the Supercomputer Toolkit [14]. The Supercomputer Toolkit is the successor to the Digital Orrery [15]. It is a small multiprocessor computer optimized for the numerical solution of systems of ordinary differential equations. The Toolkit was built as a collaboration between MIT and Hewlett-Packard. It is a family of standard software and hardware modules that can be interconnected in a variety of configurations as appropriate for particular applications. Each processor of the Toolkit is three times faster than the entire Digital Orrery, as measured by running the same computation. We used the eight-processor configuration of the Toolkit at MIT, with the new symplectic n-body map, to carry out eight 100 million year integrations of the planetary system. Each processor was used to run a separate integration with slightly different initial conditions so that we could look for exponential divergence.

The most complete of the long-term integrations of the whole Solar System is that of QTD. Particular care was taken by QTD to make their physical model accurate. They included general relativistic corrections, and a carefully crafted quadrupole approximation to account for the long-term effects due to the finite size of the Earth-Moon system. They used initial conditions and masses derived from JPL ephemeris DE102. Our physical model is the same as that of QTD except in our treatment of the effects of general relativity. General relativistic corrections can be written in Hamiltonian form, but we have not been able to integrate them analytically. Instead we used the potential approximation of Nobili and Roxburgh [16], which is easily integrated, but only approximates the relativistic corrections to the secular evolution of the shape and orientation of the orbit.

Multistep methods typically require more than a hundred steps per orbit for stability; the mapping method is stable with as few as 10 steps per orbit. The step-size for our integration was rather arbitrarily chosen to be 7.2 days; with 7.2-day steps output points are easily compared to the ephemeris of QTD. We integrated backward in time. A 22 million year integration using 3.6-day steps was carried out to check that the 7.2-day integration was sufficiently accurate.

All of the position calculations were done in pseudo-quadruple precision, in an effort to reduce the effect of roundoff error. We believe now that the use of quadruple precision was an unnecessary precaution, and needlessly doubled the computation time of our experiment. Even on this demanding task each processor evolved the Solar System at the rate of about 30 years per second. Thus our eight 100 million year integrations took a total of about 1000 hours of Toolkit time.

We have compared our integration to the reversed-time segment of the 0.5-day step-size integration of QTD, and the agreement is quite good. The maximum difference in the argument of perihelion of Mercury over the 3 million year interval is of order 0.0001 radians; for comparison, the precession of the argument of perihelion due to general relativity is about 2π radians over 3 million years. Evidently, our more approximate treatment of relativity is of little consequence. As another comparison, Table 1 lists the maximum differences between the eccentricities of the planets compared to QTD. Also listed are the differences between the integration of QTD and that of Laskar [9]. The

	$\lvert\text{Laskar} - \text{QTD}\rvert$	$\lvert\text{Toolkit} - \text{QTD}\rvert$
Mercury	0.0041	0.000018
Venus	0.0020	0.000065
Earth	0.0024	0.000059
Mars	0.0041	0.000132
Jupiter	0.0038	0.000047
Saturn	0.0081	0.000162
Uranus	0.0051	0.000008
Neptune	0.0026	0.000002
Pluto	–	0.000001

Table 1: The maximum differences in the eccentricities of the planets in the integrations of Laskar, QTD, and the Toolkit show excellent agreement among the integrations. Laskar did not include Pluto in his calculation.

table illustrates that the evolution computed with the mapping agrees quite well with the more conventional direct integration of QTD. The Toolkit integration and QTD are mutually more consistent than either is to the integration of Laskar, though it is not clear whether this discrepancy is due primarily to model differences or to the approximations used by Laskar. The model differences are actually to our benefit because they help address the important question of the sensitivity of our results to model parameters.

The Evolution of the Solar System is Chaotic

Exponential divergence of nearby trajectories is indicative of chaotic behavior. The divergence of trajectories may have both quasiperiodic and exponential components. Of course, if present, the exponential component eventually

dominates. In order to estimate the Lyapunov exponent, we extracted the exponential part of the divergence from the quasiperiodic part by the following procedure. We first converted the positions and momenta into Keplerian elements, then formed the standard shape and orientation variables $h = e \sin \varpi$, $k = e \cos \varpi$, $p = \sin i/2 \sin \Omega$, and $q = \sin i/2 \cos \Omega$, where e is the eccentricity, ϖ is the longitude of perihelion, i is the inclination, and Ω is the longitude of the ascending node. The full set of these variables for all of the planets constitutes the "full secular phase space." Distance in this space is the ordinary Euclidean distance. The quasiperiodic component is not as strong in the secular phase space as it is in the full phase space, allowing us to study the exponential part of the divergence more easily. If exponential divergence is observed in the secular phase space exponential divergence will also be present in the full phase space.

The divergence in the secular phase space between two of the 100 million year calculations is shown in Figure 1. The initial conditions for the two calculations differed by about 1 millimeter in the x coordinate of Pluto. The secular divergence has two distinct segments. In the latter segment the divergence is dominated by an exponential with a timescale of about 4 million years. The initial segment is apparently dominated by a smaller exponent with a timescale of about 12 million years.

If the system is chaotic then the divergence of individual planets will also be exponential with the same exponent as the whole system if the trajectories are followed for sufficiently long time. Over shorter intervals the divergence of individual planets can be different, and examination of the individual divergences can give insight into the mechanisms responsible for the chaotic behavior in the system.

In the inner Solar System the individual planet divergences in the secular phase space are similar to that of the full secular phase space, with two distinct segments. In the outer Solar System, over most of the 100 million years spanned by our integrations the divergences appear to be dominated by a 12 million year exponential component, with evidence of the 4 million year component appearing in only the last 5 million years. A change in the timescale of exponential divergence could occur if the system went from one region of the phase space characterized by one exponent into another region characterized by the other exponent. This possibility can be ruled out because the 4 million year component appears much later in the outer planets than in the inner planets. A viable interpretation is that there are two distinct mechanisms generating exponential divergence that are simultaneously operating. Apparently the inner planets are more sensitive indicators of the 4 million year process than are the outer planets.

The evolution of Pluto over the full 100 million years has characteristics quite similar to those found in long term integrations of the outer planets. For example, we observe the 34 million year amplitude modulation of the 3.8 million year oscillation of the argument of perihelion of Pluto [5].

Figure 2 shows the exponential divergence of the difference of positions of

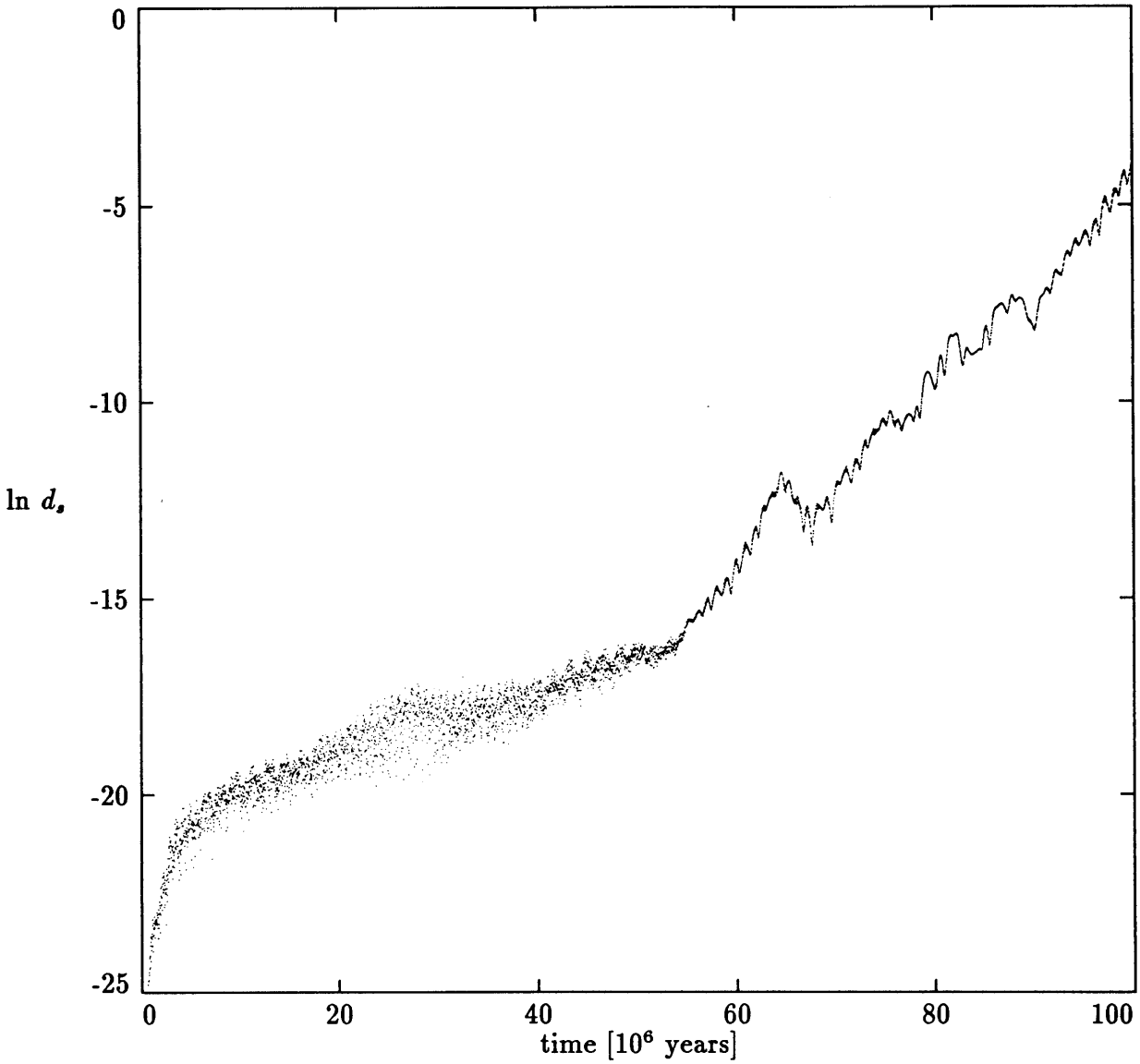

Figure 1: Exponential divergence of nearby trajectories is indicated by the average linear growth of the logarithm of the full secular phase space distance between two different runs with very slight differences in initial conditions. The segment following the initial transient has an exponential timescale of about 12 million years. The divergence is subsequently dominated by an exponential with a timescale near 4 million years.

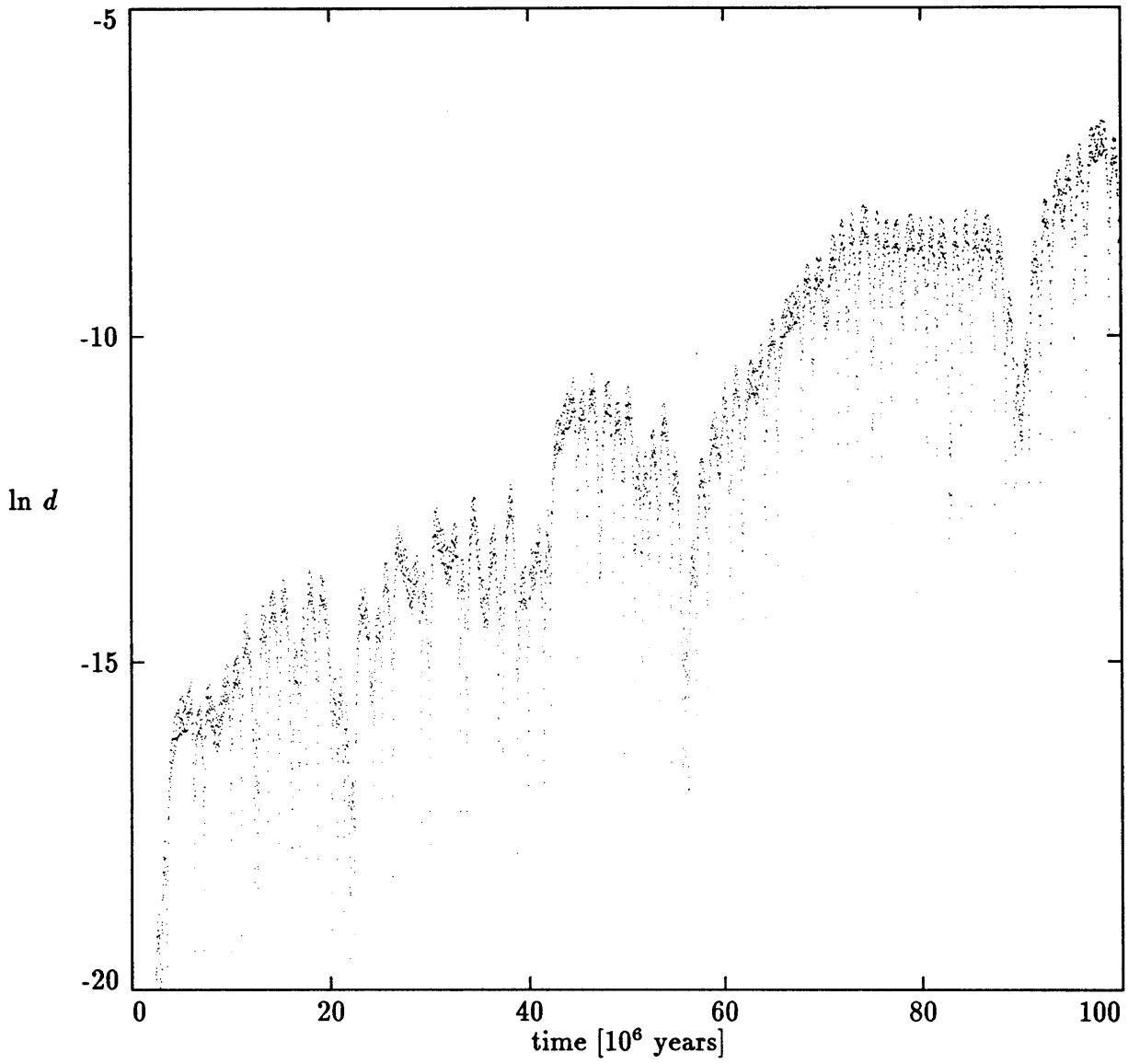

Figure 2: The divergence of the distance between the positions of Pluto in two different runs with very slight differences in initial conditions is characterized by a timescale near 12 million years.

Pluto in two of our 100 million year runs. These runs differed by 1 part in 10^{15} in one coordinate of the initial position of Mars. The plot is consistent with an exponential divergence of about 12 million years. This rate of exponential divergence is similar to that observed in the outer planets, and to the slower component observed in the full secular phase space. One interpretation is that Pluto is passively driven by chaotic behavior of the rest of the Solar System. Another interpretation is that Pluto is independently chaotic with a rate of exponential divergence which is only coincidentally similar to that of the rest of the system. We present evidence for the latter interpretation below. Considering the typical slow convergence of Lyapunov estimates, a 12 million year timescale of exponential divergence is in satisfactory agreement with the inverse Lyapunov exponent of 15 to 20 million years for Pluto reported by Sussman and Wisdom [1].

Secular Resonances

Laskar [17, 18] has found three resonance angles which alternately librate and circulate. This may be an important corroboration of the chaos as indicated by the exponential divergence. The Laskar angles are

$$\sigma_1 = (\varpi_1^\circ - \varpi_5^\circ) - (\Omega_1^\circ - \Omega_2^\circ), \tag{3}$$

$$\sigma_2 = 2(\varpi_4^\circ - \varpi_3^\circ) - (\Omega_4^\circ - \Omega_3^\circ), \tag{4}$$

$$\sigma_3 = (\varpi_4^\circ - \varpi_3^\circ) - (\Omega_4^\circ - \Omega_3^\circ). \tag{5}$$

The quantities ϖ_i° and Ω_i° are the angles of the proper modes as defined by Laskar [17]. On the basis of the alternate libration of the angles σ_2 and σ_3, which cannot simultaneously librate, Laskar concluded that the mechanism responsible for the chaotic behavior of the Solar System was resonance overlap of the corresponding secular resonances.

In our calculation, σ_1 and σ_2 also alternately circulate and librate, but σ_3 just circulates, as shown in Figure 3. Our angles track Laskar's angles for the initial segment of the computations, but they soon diverge enough so that the changes between libration and circulation occur at very different times. Such differences are consistent with the chaotic character of the evolution. The higher-order angle

$$\sigma_4 = 3(\varpi_4^\circ - \varpi_3^\circ) - 2(\Omega_4^\circ - \Omega_3^\circ) \tag{6}$$

is also incompatible with σ_2 and σ_3. In our calculation σ_4 has intervals of libration. The data presented by Laskar suggest that this is also the case in his calculation, though the angle is not explicitly mentioned. Since $1/1 < 3/2 < 2/1$ it is possible that we and Laskar are seeing different portions of a chaotic zone that spans the phase space from the 1/1 resonance to the 2/1 resonance. In addition to these angles we also find that the angle

$$\sigma_5 = (\varpi_1^\circ - \varpi_8^\circ) + (\Omega_1^\circ - \Omega_8^\circ) \tag{7}$$

alternates between circulation and libration.

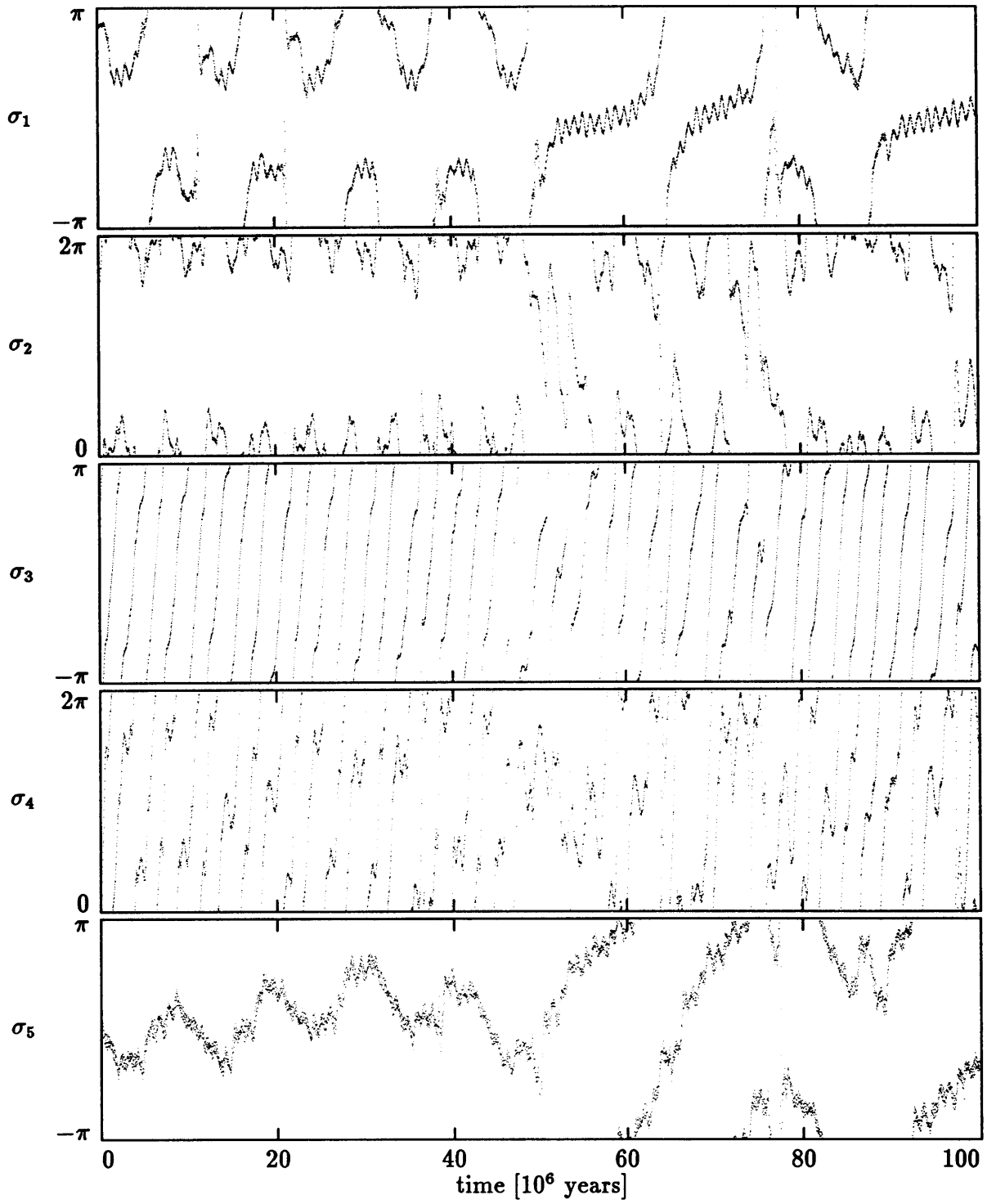

Figure 3: In our calculation two of the angles of Laskar, σ_1 and σ_2, alternately circulate and librate. Laskar's third angle, σ_3, circulates. The higher order angle σ_4, which is incompatible with both σ_2 and σ_3, has intervals of libration. The angle σ_5 also alternately circulates and librates.

Though our calculations are completely consistent with those of Laskar, we are not fully convinced that his proposed mechanism accounts for the observed chaotic behavior of the Solar System. First, it is not clear that the alternate libration and circulation of σ_2 is indicative of a dynamically significant chaotic separatrix. In Figure 4 we show a polar plot of the amplitude and phase of the combination of proper modes corresponding to σ_1 and σ_2. Our polar plot of σ_1 is similar to that shown in Laskar [17]. However, our polar plot of σ_2 differs from that shown in Laskar [17] and Laskar, Quinn, and Tremaine [9]. For the polar amplitude they use just the amplitude of eccentricity mode 4, which is never close to zero. The amplitude corresponding to σ_2 has additional factors, some of which approach zero. Using just the amplitude of eccentricity mode 4 gives an artificial impression of a transition from libration to circulation. Our plot, which includes all the factors in the amplitude, does not give a clear indication of a chaotic separatrix. Rather, it gives the impression of a complex high-dimensional trajectory projected onto a plane (see Figure 4). The alternation of libration and circulation of the polar angle may just be an artifact of the origin being in the midst of this complex projected trajectory. A simple integrable model for motion near the 2/1 commensurability in the planar-elliptic restricted three-body problem [19, 20] illustrates how the alternate circulation and libration of an angle can be misleading. There the resonance angle corresponding to the largest term in the disturbing function can show alternate circulation and libration, even though there is no chaotic behavior. The phenomenon can be eliminated by an appropriate change of variables. The polar plot corresponding to σ_1 is more like that expected of an angle associated with a chaotic mechanism. It not only alternately circulates and librates, but as it circulates it loiters around an apparent unstable equilibrium. There is also an excluded region near the center of the plot.

Furthermore, there are too many unrelated angles which alternately circulate and librate. It might be expected that only one incompatible set of angles would show alternate circulation and libration. However, we see the phenomenon in angles involving unrelated modes. Eccentricity and inclination modes 3 and 4 are involved in σ_2, σ_3, and σ_4; eccentricity modes 1, 5, and 8, and inclination modes 1, 2, and 8 are involved in σ_1 and σ_5. The two sets of modes are disjoint, and yet there are correlations in the behavior of angles associated with unrelated sets of modes. Most striking is the transition in behavior in four of the angles near an integration time of 50 million years. This suggests that a single mechanism is driving all of the angles. If the mechanism is associated with one of the angles presented, we feel the most convincing is σ_1. It is also possible that the mechanism generating the chaos is unrelated to all of these angles, and that they are all just sensitive indicators of chaotic irregularity of the underlying system trajectory.

We have not identified any other angles whose motion is suggestive of a dynamical mechanism. The LONGSTOP team speculated [21] that the angle $2\varpi_5^\circ - 2\varpi_7^\circ + \Omega_7^\circ - \Omega_8^\circ$ might alternately circulate and librate, and thereby provide evidence of chaotic behavior of the outer planets. We find that this

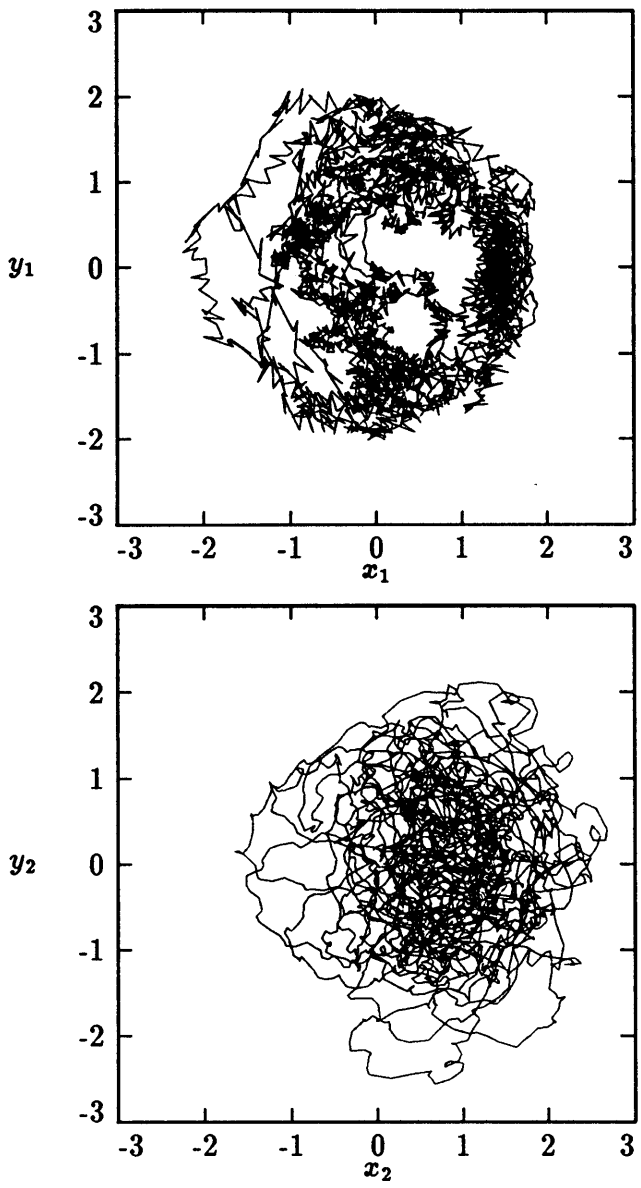

Figure 4: The variables are $x_i = A_i \cos \sigma_i$ and $y_i = A_i \sin \sigma_i$, where A_i are the amplitudes of the combination of the Laskar proper modes corresponding to the angles σ_i. The plot corresponding to σ_2 gives the impression of a complex high-dimensional trajectory projected onto a plane. The plot corresponding to σ_1 shows some indication of a dynamically significant chaotic separatrix. There is a suggestion of an unstable equilibrium and associated asymptotic trajectories, and there is a region near the center that is avoided.

angle rotates uniformly with a period somewhat greater than 50 million years. This angle is one of 30 angles presented by the LONGSTOP team in their "numerology table." We find all 30 angles rotate uniformly; there is no hint of chaotic irregularity in the behavior of any of the angles.

That particular secular resonances are responsible for the chaotic behavior of the Solar System could be established by an analytical demonstration that the resonances involved are sufficiently strong and close to give resonance overlap. At the moment, however, we feel no dynamical mechanism for the observed chaotic behavior of the Solar System has been clearly demonstrated.

Chaotic Evolution of the Jovian Planets

Despite the growing number of long-term integrations of the outer planets, to date no integration has directly tested whether the massive planets themselves evolve chaotically. In our 845 million year integration of the outer planets [1], the orbital elements of Neptune appeared to have discrete line spectra, in marked contrast to the clearly broad-band character of the spectrum of Pluto. On the basis of this spectral evidence we dismissed the possibility of chaotic behavior in the Jovian planets. To be thorough, we have now carried out a new billion year evolution of the outer planets, using the mapping method, with a slightly perturbed initial position of Neptune. We found, to our surprise, that the subsequent evolution of the Jovian planets diverged exponentially from the first calculation, with a timescale of exponential divergence of only 5 million years!

Our initial reaction was that there must be something wrong with our method of integration. To check this we carried out two new integrations of the outer planets spanning 100 million years using the traditional linear multistep Stormer predictor, the same integrator used in the Digital Orrery integrations. The initial conditions and masses were the same as in the Digital Orrery integrations. The integrations were carried out in ordinary IEEE double precision (64 bits) with a step-size of 32.7 days, the same step-size used in the 845 million year Digital Orrery integrations. We found that the trajectories of the Jovian planets diverged exponentially with a timescale of about 19 million years. Apparently, we were misled by the spectral evidence.

To further check that this result did not depend on either the step-size or the precision of the calculation we carried out four more integrations of the outer planets using the Stormer predictor. Each spanned more than 400 million years. In these integrations the accumulation of position was carried out in pseudo-quadruple precision, as in the Digital Orrery integrations. One pair used a step-size of 32.7 days, the special Digital Orrery step-size; the other pair used an arbitrarily chosen step-size of 28 days. The initial conditions were the same as in our earlier outer planet integrations. In one run of each pair the initial position of Neptune was perturbed by 7.5mm. The energy errors again grew linearly with time, with slopes between that of our 210 million year Digital Orrery integration and that of our 845 million year integration. Both pairs of runs gave remarkably consistent results. The secular divergence of the

Jupiters in the 28-day runs is shown in Figure 5. The Jovian planets diverged exponentially with a timescale somewhat longer than 20 million years.

The map and Stormer calculations both indicate that the motion of the Jovian planets is chaotic. However, they are discrepant in the estimate of the exponential timescale. We believe this is a result of the fact that the trajectory computed by the mapping method is not exactly the same as the trajectory determined by the initial conditions. The mapping differs from the actual n-body dynamics by the addition of extra high-frequency terms. The averaging principle suggests that these high-frequency terms do not contribute to the long-term evolution, and the close agreement of our results with those of Laskar once again confirms the validity of the use of averaging. However, the initial conditions used in an averaged system should properly take into account the presence of the extra high-frequency components in the unaveraged system. The use of the same initial conditions in the averaged and unaveraged systems corresponds to slightly different initial conditions for the long-term evolutions. With the mapping we do not yet know how to properly adjust the initial conditions to account for the extra high-frequency terms. In our outer-planet integrations these slight uncontrolled adjustments of effective initial conditions appear to yield different estimates of the Lyapunov exponent.

To investigate this further we carried out 8 integrations of the outer planets using the map with different step-sizes, ranging from 0.3 years to 1 year. Changing the step-size changes the high-frequency components, and slightly changes the effective initial condition for the long-term evolution. Each integration spanned about 300 million years. We found that the measured divergence timescale varied from about 3 million years to about 30 million years. The dispersion in the estimates of the Lyapunov exponent are much larger than the dispersion observed in the Stormer runs. The Lyapunov exponent was not obviously correlated with step-size, in particular the estimate of the Lyapunov exponent was not monotonic with step-size. In one of these runs, with step-size near 0.617979 years, the motion of the outer planets was clearly quasiperiodic; the secular phase space divergence did not grow exponentially. The results suggest that the Lyapunov exponent for the Jovian planets is not a simple function of the initial conditions. Most nearby initial conditions lead to exponential divergence (most with a shorter timescale for exponential divergence than that obtained with the Stormer integrations), but there are also nearby initial conditions that do not give chaotic behavior.

With each outer planet integration we ran a pair of massless Plutos, with initial positions differing by about 1cm. A remarkable result is that the exponential divergence of Plutos always has a timescale between 10 and 20 million years, independent of how chaotically the Jovian planets behave. This is true even in the most extreme runs, where the Jovian planets were quasiperiodic, and where the Jovian planets diverged with a timescale of 3 million years. This clearly demonstrates that the mechanism generating the chaotic behavior in the motion of Pluto is extremely robust, and independent of whether the rest of the system is chaotic.

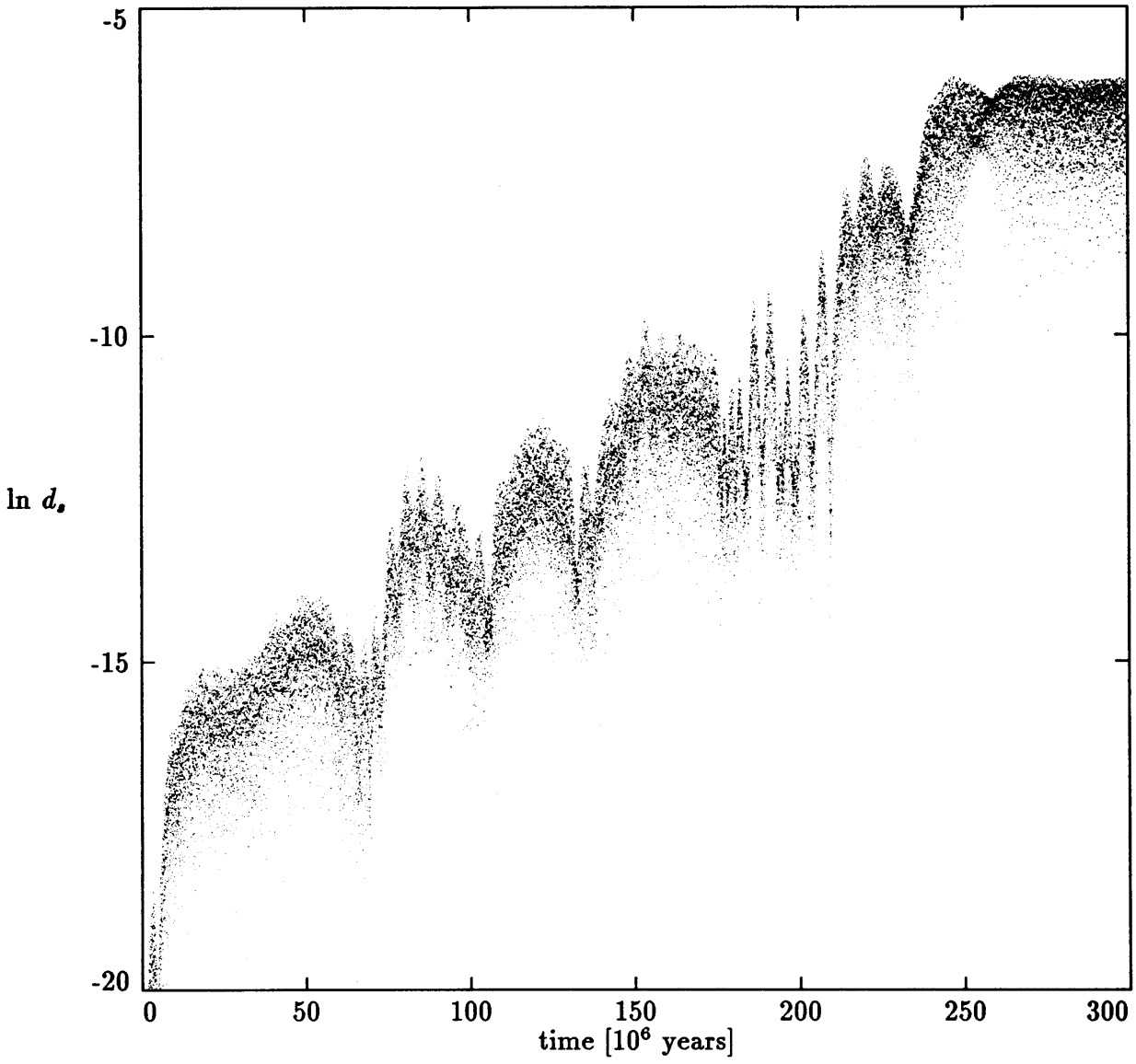

Figure 5: The secular phase space divergence of the trajectories of Jupiter in the 28 day Stormer integrations show a timescale for exponential divergence that is somewhat longer than 20 million years. The divergence saturates after about 250 million years at a rather small value, perhaps indicating a hidden constraint on the trajectories.

Numerical Chaos?

The fact that almost all long-term integrations of the Solar System give exponential divergence of trajectories with a timescale in the range of 3-30 million years in physically quite different models is very striking and unsettling. Another unsettling feature of the chaotic behavior we observe in long-term planetary integrations is that nothing dramatic happens. This is compounded by the fact that in no case have we unambiguously identified the mechanism responsible for the chaotic behavior. This lack of mechanism and lack of obvious irregular behavior is in marked contrast to the clearly understood mechanisms and irregular character observed in other examples of chaotic behavior in the solar system, for example, asteroids on chaotic trajectories near commensurabilities [11, 12, 22], the chaotic tumbling of Hyperion [23] and other irregularly shaped satellites [24], and the chaotic motion of the Uranian satellites near commensurabilities [25, 26, 27].

Perhaps the exponential divergence is a numerical artifact? The detailed agreement of the billion year evolution of Pluto using different integrators is impressive evidence against this perverse possibility. Furthermore, the detailed agreement between our 100 million year Solar System integration and that of Laskar is particularly convincing, because of the radically different methods used.

To further convince ourselves that not all long-term integrations are subject to some universal numerical instability, which yields a spurious exponential divergence, we have carried out a control integration. We have integrated the outer planets without Uranus for about 250 million years, with the mapping method. Over that period we see no sign of exponential divergence of the remaining Jovian planets. This integration, together with the isolated quasiperiodic integration mentioned above, shows that numerical models of planetary systems can, in fact, show quasiperiodic behavior on a several hundred million year time-scale. Long-term integrations do not always give positive Lyapunov exponents.

Altogether, the evidence for the chaotic behavior in these long-term planetary integrations is very convincing, but there remains the logical possibility that the exponential divergence is a subtle numerical artifact. To positively conclude that the chaos observed in these long-term planetary integrations is not a result of numerical artifacts requires an unambiguous identification of a physical mechanism and an analytic evaluation to determine that the mechanism actually accounts for the observed chaos.

Conclusions and Speculations

Our 100 million year integration of the entire Solar System indicates that the Solar System is chaotic with a timescale for exponential divergence of about 4 million years. The fact that we find similar behavior in all respects to the calculation of Laskar strongly supports the conclusion that the Solar System is chaotic. That we and Laskar have carried out different kinds of numerical experiments, with slightly different masses, slightly different initial

conditions, and slightly different physics, shows that the chaotic character of the Solar System is not sensitively dependent on the precise model or numerical methods.

Our experiments indicate that the Jovian planets by themselves behave chaotically for most initial conditions near our reference system, though our estimates of the Lyapunov exponent have rather large dispersion.

All of our estimates of the Lyapunov exponent of Pluto give approximately the same divergence timescale of 10-20 million years, with different methods of integration, different planetary masses, different initial conditions, and even independently of whether the rest of the system is behaving chaotically. Our earlier result that the evolution of Pluto is chaotic is thus multiply confirmed.

We will not fully understand the consequences of the observed chaotic evolution of the Solar System until we clearly understand the dynamical mechanisms responsible for it. Though in our calculation the behavior of the secular resonance angles is consistent with those in Laskar's calculation, the identification of resonance overlap of particular secular resonances as the mechanism generating the chaotic behavior of the Solar System is not unambiguously demonstrated. Our numerical experiments suggest that there are at least two independent mechanisms generating chaotic behavior. One mechanism generates chaos in the Jovian subsystem, and Pluto is independently chaotic in the field of the Jovian planets. Yet another mechanism is suggested by the simultaneous presence of two different exponential timescales in our full Solar System integrations. Secular resonances among the inner planets may be driving the faster timescale, as Laskar suggested. However, the most convincing of the secular resonances involve Mercury. Both Mercury and Pluto have high eccentricity and inclination, which strongly couples the eccentricity and inclination secular subsystems. Perhaps one of the mechanisms generating the chaos originates with Mercury, and is similar to the mechanism generating the chaos in the motion of Pluto?

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