MASSACHUSETTS INSTITUTE OF TECHNOLOGY

PROJECT MAC

Artificial Intelligence
Memo. No. 159

MAC-M-371
April 1968

NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS BY SPLINE FUNCTIONS

Jayant M. Shah

A numerical method for solving linear, two-dimensional elliptic boundary value problems is presented. The method is essentially the Ritz procedure which uses polynomial spline functions to approximate the exact solution. The spline functions are constructed by defining a polynomial function over each of a set of disjoint subdomains and imposing certain compatibility conditions along common boundaries between subdomains. The main advantage of the method is that it does not even require the continuity of the spline functions across the boundaries between subdomains. Therefore it is easy to construct classes of spline functions which will produce any specified rate of convergence.

1. INTRODUCTION

During the last few years polynomial splines have been used to obtain numerical solutions of specific elliptic boundary value problems [2,3,4]. This paper presents a general formulation of this type for solving two-dimensional elliptic boundary value problems. The method is essentially the Ritz procedure applied to a finite dimensional space of polynomial spline functions. The formulation can be extended to higher dimensional cases and to simultaneous systems.

We denote by G a bounded open domain in the Euclidean plane with boundary ∂G and closure $\bar{G}$. For a function $u \in C^j(\bar{G})$ we denote by $|||u|||_{j,G}$ the maximum norm $\max_{|\alpha|\leq j} \max_{\bar{x}\in\bar{G}} |D^\alpha u(\bar{x})|$ and denote by $||u||_{j,G}$ the norm $\left[\int_G \sum_{|\alpha|\leq j} |D^\alpha u(\bar{x})|^2 d\bar{x}\right]^{1/2}$. Here $\alpha = (\alpha_1, \alpha_2)$ is the multiple index of length $|\alpha| = \alpha_1 + \alpha_2$ and D^α is the corresponding partial derivative. We denote by $|u|_{j,G}$ the semi-norm $\left(\int_G \sum_{|\alpha|=j} |D^\alpha u|^2 d\bar{x}\right)^{1/2}$. Let $L = \sum_{|\alpha|\leq 2m} b_\alpha(\bar{x})D^\alpha$ be a real strongly elliptic operator of order 2m defined in $\bar{G}$. Suppose that $b_\alpha \in C^\infty(\bar{G})$ and G is of class C^∞. Let the bilinear form associated with the operator L be given by

$$(1.1) \qquad B(u,v) = \sum_{\substack{|\alpha|\leq m \\ |\beta|\leq m}} \int_G \left\{ (D^\alpha u)(b_{\alpha\beta} D^\beta v) \right\} d\bar{x}$$

where $u, v \in C^{\infty}(\overline{G})$. Then $L = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} (-1)^{|\alpha|} D^{\alpha}(b_{\alpha\beta} D^{\beta})$. We assume that $B(\cdot,\cdot)$ satisfies the Garding's inequality in the following form:

$$(1.2) \qquad |B(\varphi,\varphi)| \geq C|\varphi|^2_{m,G} \quad \text{for all } \varphi \in C^{\infty}(\overline{G}')$$

where G' is any open subset of G.

For a given $f \in C^{\infty}(\overline{G})$, let $\overline{u} \in C_0^{\infty}(\overline{G})$ be the infinitely differentiable solution satisfying the equation $L\overline{u} = f$ and satisfying zero Dirichlet data on the boundary. Then

$$(1.3) \qquad B(\varphi, \overline{u}) = \int_G \varphi f d\overline{x} \quad \forall \varphi \in C_0^{\infty}(G)$$

2. TWO-DIMENSIONAL SPLINE FUNCTIONS

Subdivide the domain $\overline{G}$ by a grid $\vdash$ into a finite number of subdomains such that each subdomain is a polygon with infinitely differentiable curvilinear sides. We call any such subdomain a cell. Let $\{S_i\}$ be the set of grid lines forming the sides of cells. We require that the interior of any grid line S_i

must not contain a vertex. Let $\sup_i$ (Length of S_i) = h. We denote by $\{G_i\}$ the disjoint collection of interiors of the cells and let $G_0 = \bigcup_i G_i$.

Assume that each grid line is given by an equation of the form

$$\text{(1.4)} \qquad \begin{aligned} x &= f_i\ (\xi_i) \\ y &= g_i\ (\xi_i) \end{aligned}$$

where ξ_i is the distance along the grid line S_i measured from one of the vertices which it connects. The unit tangent vector $\vec{t}_i = (t_1, t_2)$ at a point on the grid line and the unit normal $\vec{n}_i = (-t_2, t_1)$ are defined by the equation:

$$\text{(1.5)} \qquad t_1 = \frac{df_i/d\xi_i}{\left[\left(\frac{df_i}{d\xi_i} \right)^2 + \left(\frac{dg_i}{d\xi_i} \right)^2 \right]^{1/2}}$$

$$t_2 = \frac{dg_i/d\xi_i}{\left[\left(\frac{df_i}{d\xi_i} \right)^2 + \left(\frac{dg_i}{d\xi_i} \right)^2 \right]^{1/2}}$$

If $u^{(i)} \in C^\infty(\overline{G}_i)$, let $D^{(j)}_{n,r}\, u^{(i)}$ denote the r-th derivative of $u^{(i)}$ in the direction of the normal $\vec{n}_i$, defined if S_j is one of the sides of G_i. For every cell, we define the following quantities:

$$(1.6) \qquad u^{ij}_{kr} = \frac{1}{h^{k+1}} \int_{S_j} \xi_j^k \, D^{(j)}_{n,r} \, u^{(i)} \, d\xi_j \qquad \text{defined for all } k \text{ and } r \text{ whenever } 0 \le r \le m-1 \text{ and } 0 \le k \le p+r-(m-1)$$

where p is some fixed integer ≥ 0 and independent of h. Also if w is a vertex of a cell G_i, define $u^{i,w}_{\alpha} = D_{\alpha} u^{(i)}(w)$ for all α such that $0 \le |\alpha| \le m-2$.

We define a class $\mathcal{F}(h,p)$ of functions with domain G as follows: Let (x_i, y_i) denote the local coordinates of a cell G_i, obtained by translation of the origin to a point in G_i. If $u \in \mathcal{F}(h,p)$, then $u^{(i)}$ = u restricted to G_i is given by an equation of the form

$$(1.7) \qquad u^{(i)} = a_{oo} + a_{1o}\, x_i + a_{o1}\, y_i + \ldots + a_{o,n_i}\, y_i^{n_i}$$

such that the following compatibility conditions are satisfied:

(i) For all k and r and whenever defined

(1.8) $$u^{ij}_{kr} = u^{i'j}_{kr} \quad \text{if } S_j = \partial G_i \cap \partial G_{i'}$$

and $$u^{ij}_{kr} = 0 \quad \text{if } S_j = \partial G \cap \partial G_i$$

(ii) At a vertex w, for all $|\alpha| \leq m-2$

(1.9) $$u^{i,w}_{\alpha} = u^{i',w}_{\alpha} \quad \text{if } w \text{ belongs to both } G_i \text{ and } G_{i'}$$,

and $$u^{i,w}_{\alpha} = 0 \quad \text{if the } w \epsilon \partial G$$

We also require that

(1.10) $$n_i \geq m + p$$

Now we impose the following conditions on the class of admissible grids:

First we introduce non-dimensional coordinates $X_i = x_i/h$, $Y_i = y_i/h$, $\Xi_i = \xi_i/h$ and quantities $A_{jk} = a_{jk}h^{j+k}$, $U^{ij}_{kr} = u^{ij}_{kr}h^{r}$ and $U^{i,w}_{\alpha} = u^{i,w}_{\alpha}h^{|\alpha|}$ so that

(1.11) $$u^{(i)} = A_{oo} + A_{1o}X_i + \ldots A_{on_i}Y_i^{n_i}$$

$$(1.12)\qquad U^{ij}_{kr} = \int_{S_j} \Xi^k_j \, (h^r D^{(j)}_{n,r}) u^{(i)} d\Xi j, \quad U^{i,w}_{\alpha} = (h^{|\alpha|} D_{\alpha}) u^i (w)$$

For a given cell G_i, let V_i be the vector space spanned by the quantities U^{ij}_{kr} and $U^{i,w}_{\alpha}$ and let A_i be the vector space spanned by the coefficients A_{jk}. Then we have a linear transformation

$$(1.13)\qquad T_i : A_i \to V_i$$

We choose n_i sufficiently large so that T_i is onto. Now let K_i be the kernel of T_i and let $\overline{A}_i$ be the complement of K_i. Then T_i induces an isomorphism

$$(1.14)\qquad \overline{T}_i : \overline{A}_i \simeq V_i$$

We require that : (a) Values of n_i are bounded uniformly with respect to all admissible grids and the absolute values of the eigenvalues of $\overline{T}_i$ are bounded away from zero uniformly with respect to all admissible grids. (b) There exists a positive constant δ_1 such that for any admissible grid,

$$\inf_i \frac{\text{length of } S_i}{h} \geq \delta_1$$

(c) There exists a positive constant δ_2 such that for every admissible grid, and for every cell, there exists a point (x_o,y_o) in that cell such that the square $\{(x,y)||x-x_o|\le\delta_2 h, |y-y_o|\le\delta_2 h\}$ is contained in the cell. (d) Modulo translation, rotation and scale factor, grid lines S_i are restricted to a finite number of shapes. (e) There exists a constant C_o such that the number of cells that any straight line through G intersects is less than C_o/h.

If the cells are triangular or rectangular in shape with straight sides and if they satisfy conditions (b) and (c), then it is possible to choose a single integer n for n_i in Eq. (1.7) such that condition (a) is satisfied for all such cells. To show this for triangular cells, consider a cell G_i in the form of an isosceles right-angled triangle with hypothenuse h units long and with origin of the local coordinate system at one of the vertices. Since the transformation T_i for this particular cell does not depend upon h, we can find an integer n for n_i in Eq. (1.7) such that condition (a) is satisfied for all values of h. Now consider any other triangular cell $G_{i'}$ which satisfies conditions (b) and (c). Set $n_{i'} = n$. Without loss of generality we can assume that $G_{i'}$ has the origin of the local coordinate system at one of its vertices. There exists a non-singular linear transformation S mapping G_i onto $G_{i'}$. Now specification of the quantities $u_\alpha^{i,w}$ and u_{kr}^{ij} (if defined) implies specification of all quantities of the form

$$\frac{1}{h^{k+1}} \int_{S_j} \xi_j^k \, D_{n,r}^{(j)} \, D_{t,r'}^{(j)} \, u^{(i)} d\xi_i \qquad \begin{array}{l} 0 \le r+r' \le m-1 \\ r' > 0 \\ 0 \le k \le p+(r+r')-(m-1) \end{array}$$

where $D_{t,r'}^{(j)} u^{(i)}$ denotes the r'-th derivative of $u^{(i)}$ in the direction of the tangent, defined if S_j is one of the sides of G_i.

Therefore S induces an isomorphism

$$(1.17) \qquad S_2 : V_i \simeq V_{i'}$$

Also S induces an isomorphism

$$(1.18) \qquad S_1 : A_i \simeq A_{i'}$$

Therefore we have a commutative diagram

$$(1.19) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K_i & \longrightarrow & A & \xrightarrow{T_i} & V_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow S_1 & & \downarrow S_2 & & \\ 0 & \longrightarrow & K_{i'} & \longrightarrow & A_{i'} & \xrightarrow[T_i]{} & V_{i'} & \longrightarrow & 0 \end{array}$$

This, in turn, implies a commutative diagram

(1.20)
$$\begin{array}{ccc} \bar{A}_i & \xrightarrow{\bar{T}_i} & V_i \\ \downarrow S_1 & & \downarrow S_2 \\ \bar{A}_{i'} & \xrightarrow{\bar{T}_{i'}} & V_{i'} \end{array}$$

Therefore $\bar{T}_{i'}$ is nonsingular. Now we claim that condition (a) is satisfied. For, if not, there exists a sequence of triangular cells $\{G_k\}$ such that the corresponding sequence of the lowest absolute eigenvalues of the transformations $\bar{T}_k$ tends to zero. But any triangular cell is specified by six quantities, namely the non-dimensional coordinates of its vertices. Therefore the sequence of triangles can be embedded as a closed subspace lying in the closed annulus in $\mathbb{R}^6$, lying between the unit sphere and the sphere of radius δ_1. Since the annulus is compact, the sequence G_k has an accumulation point. By continuity, the triangular cell G_{ko} corresponding to the accumulation point satisfies conditions (b) and (c) and has singular $\bar{T}_{ko}$ which contradicts (1.20).

In the same fashion, we can choose a single integer in Eq. (1.7) which will satisfy condition (a) for all rectangular cells provided that conditions (b) and (c) are satisfied.

For practical applications, it suffices to segment a domain using triangular and rectangular cells as well as a finite number of special shapes to handle a curved boundary.

The vector space $\mathfrak{F}(\mathfrak{h},p)$ is constructed as follows. For each cell G_i, we have an isomorphism $A_i \simeq V_i \oplus K_i$. Then a function $u \in \mathfrak{F}(\mathfrak{h},p)$ is defined by specifying the quantities

$$u^j_{kr} = u^{ij}_{kr} = u^{i'j}_{kr} \qquad \text{for each grid line not in } \partial G$$

$$u^w_\alpha = u^{i,w}_\alpha = u^{i',w}_\alpha = \ldots \qquad \text{for each vertex not in } \partial G$$

and an element of $K^{(i)}$ for each cell G_i.

We require that

$$u^j_{kr} = 0 \text{ if } S_j \text{ is in } \partial G$$

and $u^{i,w}_\alpha = 0$ if w is in ∂G

We extend the (semi) norms $|\cdot|, ||\cdot||, |||\cdot|||$ and the bilinear form $B(\cdot,\cdot)$ to $\mathcal{F}(h,p)$ by restricting the integrals to G_0.

Let V be a vector space spanned by the quantities u^j_{kr}, u^w_α (S_j and w not in ∂G) with the basis $\{e_i\}$. Let $K = \bigoplus_i K_i$ with basis $\{k_i\}$. Let $W = V \oplus K$. Then we have an isomorphism $I : \mathcal{F}(h,p) \cong W$. The images of e_i and k_j under I form a basis for $\mathcal{F}(h,p)$. Denote this basis by $\{u_i\}$

3. FORMULATION

We seek a solution $\hat{u} \varepsilon \mathcal{F}(h,p)$ such that the equation

$$(1.21) \qquad B(\varphi,\hat{u}) = \int_{G_0} \varphi f d\bar{x}$$

is satisfied for all $\varphi \varepsilon \mathcal{F}(h,p)$. We show that Eq. (1.21) has a unique solution and that $D^\alpha(\hat{u}-\bar{u}) = O(h^{p+1-\gamma}) \qquad$ for $0 \leq |\alpha| \leq m-1$.

For a numerical solution of Eq. (1.21) we reduce it to an equivalent matrix equation as follows. We want to find $\hat{v} \varepsilon W$ such that $\hat{u} = I^{-1}(\hat{v})$ satisfies Eq. (1.21). Carrying out the integrations, $B(\varphi,\hat{u}) = (v,\bar{B}\,\hat{v})$ where $v = I(\varphi)$,

(,) is the inner product and $\overline{B}$ is a matrix $(\overline{B}_{ij})$ where $\overline{B}_{ij} = B(u_i,u_j)$. Also $\int_\omega f d\overline{x} = (v,F)$ where $F \epsilon V$ and its i-th component is given by $\int u_i f d\overline{x}$. Therefore Eq. (1.21) is equivalent to the matrix equation:

$$(1.22) \qquad \overline{B}\,\hat{v} = F$$

If within each cell, we approximate the coefficients $b_{\alpha\beta}$ and the function f by polynomials use of quadrature formulas for integration of polynomials over simplexes [5] simplifies the computation of $\overline{B}$. The computation is carried out by computing $B(u_i,u_j)$ cell by cell.

4. CONVERGENCE

In the following C_1, C_2, ... denote positive numbers, the choice of which does not depend upon $\mathfrak{h}$.

Lemma 1: Let $u \epsilon \mathfrak{F}(\mathfrak{h},p)$. Let S_j be a grid line contained in $\partial G_i \cap \partial G_{i'}$, and let u^i = u restricted to G_i and $u^{i'}$ = u restricted to $G_{i'}$. Then for $|\alpha| \leq m-1$

$$(1.23) \qquad \sup_{(x,y)\epsilon S_j} |D_\alpha u^i(x,y) - D_\alpha u^{i'}(x,y)| \leq C_1 h^{m-1-|\alpha|}(|u|_{m,G_i} + |u|_{m,G_{i'}})$$

If $S_j = \partial G_i \cap \partial G$, then for $|\alpha| \le m-1$

(1.24) $$\sup_{(x,y)\in S_j} |D_\alpha u^i(x,y)| \le C_1 h^{m-1-|\alpha|} |u|_{m,G_i}$$

Proof: Suppose that S_j connects vertices w_1 and w_2. Without loss of generality we can assume that w_1 is the origin and that the x-axis passes through w_2. Let λ_j be the length of S_j: Now consider a subspace $\mathfrak{a}_o$ of the vector space A_i spanned by $\{A_{oo}, A_{1o}\ A_{o1}, \dots A_{o,m-1}\}$ and a subspace $\mathfrak{b}_o$ of V_i spanned by $\{U_\alpha^{i,w_1}\}_{|\alpha|\le m-2}$, $\{U_\alpha^{i,w_2}\}_{|\alpha|\le m-2}$ and $U_{o,m-1}^{ij}$. (If m = 1, then $\mathfrak{b}_o$ is one-dimensional, spanned by $U_{o,o}^{ij}$). Let $j : \mathfrak{a}_o \to A_i$ be the inclusion map and $p : V_i \to \mathfrak{b}_o$ be the projection map. Then it is easy to check that the map $\tau_i : \mathfrak{a}_o \to \mathfrak{b}_o$ given by the composition $p\circ\tau_i\circ j$ is an isomorphism and therefore $\mathfrak{a}_o \subset \bar{A}_i$. Moreover since grid lines are restricted to a finite number of shapes and $\lambda_j/h \in [\delta_1, 1]$, it follows that the absolute values of the eigenvalues of τ_i are bounded away from zero uniformly with repect to $\mathfrak{h}$.

Let $\mathfrak{b}_1$ be the complement of $\mathfrak{b}_o$ in V_i and $\mathfrak{a}_1$ be the complement of $\mathfrak{a}_o$ in $\bar{A}_i$. Let $\mathfrak{a}_o', \mathfrak{a}_1', \mathfrak{b}_o', \mathfrak{b}_1'$ be the corresponding spaces defined for the cell $G_{i'}$.

Let $u \in \mathfrak{F}(\mathfrak{h}, p)$ and let $w \in W$ and $k \in K$ be the vectors corresponding to u.

Let the projection of w onto V_i be v and let $v = v_o \oplus v_1 \in \mathfrak{b}_o \oplus \mathfrak{b}_1$ where $v_o \in \mathfrak{b}_o$ and $v_1 \in \mathfrak{b}_1$. Let $a_o = \tau_i^{-1}(v_o)$ and let the unique polynomial corresponding to a_o be $P = \beta_{oo} + \beta_{10} x_i + \ldots + \beta_{o,m-1} Y_i^{m-1}$. Similarly, the corresponding vector $v_o' \in \mathfrak{b}_o'$ determines a unique polynomial $P' = \beta_{oo}' + \beta_{10}' x_i \ldots + \beta_{o,m-1}' Y_i^{m-1}$. But $v_o = v_o'$. Therefore $P = P'$.

Let $d = \overline{T}_i^{-1}((o \oplus v_1))$ and $d = (d_o \oplus d_1) \in \mathfrak{A}_o \oplus \mathfrak{A}_1$ where $d_o \in \mathfrak{A}_o$ and $d_1 \in \mathfrak{A}_1$.

Let $\tau_q^* = \overline{T}_i$ restricted to $\mathfrak{A}_q$, $q = o,1$. Then $p \circ \tau_o^* + p \circ \tau_1^*$ is the zero map and $p \circ \tau_o^* = \tau_i$. Therefore $d_o = -\tau_i^{-1} (\tau_1^* (d_1))$ which implies that if $d_o = (\gamma_{oo}, \gamma_{1o}, \ldots, \gamma_{o,m-1})$ and $d_1 = (\gamma_{m,o}, \ldots, \gamma_{o,n_i})$. Then $|\gamma_\alpha| \le C_2 \max_{|\beta| \le m} |\gamma_\beta|$ for $|\alpha| < m$.

We define the following norms:

For $q = (q_{oo}, q_{1o}, \ldots, q_{o,m-1}) \in \mathfrak{A}_o$ define the norm $||q|| = \max_{i,j} |q_{ij}|$;

For $r = (r_{oo}, r_{1o}, \ldots, r_{o,n_i}) \in \mathfrak{A}_1$ define the norm $||r|| = \max_{(i,j)} |r_{ij}|$;

For $s = (s_{m,o} \ldots s_{o,n_i}) \in K_i$, define the norm $||s|| = \max_{(i,j)} |s_{ij}|$

For $r \oplus s = z \in \mathcal{Q}_1 \oplus K_i$ define the norm $||z|| = ||r|| + ||s||$.

Then $||d_o|| \le C_2 ||d_1||$. Similarly $||d_o'|| \le C_2 ||d_1'||$ for the cell $G_{i'}$. Now

$\overline{T}_i\{((a_o + d_o) \oplus d_1)\} = v$. Let the projection of k onto K_i be the vector

$k_i = (k_{m,o}, \ldots k_{o,n_i})$ and let $d_1 \oplus k_i = \ell_i$ and $d_1' \oplus k_1' = \ell_i'$. Then

$$u^{(i)} = (\beta_{oo}+\gamma_{oo})+\ldots+(\beta_{o,m-1}+\gamma_{o,m-1})Y_i^{m-1}+(k_{m,o}+\gamma_{m,o})X_i^m+\ldots+(k_{o,n_i}+\gamma_{o,n_i})Y_i^{n_i}$$

$$u^{(i')} = (\beta'_{oo}+\gamma'_{oo})+\ldots+(\beta'_{o,m-1}+\gamma'_{o,m-1})Y_i^{m-1}+(k'_{m,o}+\gamma'_{m,o})X_i^m+\ldots+(k'_{o,n_i}+\gamma'_{o,n_i})Y_i^{n_i'}$$

$$u^{(i)}-u^{(i')} = (\gamma_{oo}-\gamma'_{oo})+\ldots+(\gamma_{o,m-1}-\gamma'_{o,m-1})Y_i^{m-1}+(k_{m,o}+\gamma_{m,o}-k'_{m,o}-\gamma'_{m,o})X_i^m\ldots$$

Therefore $\sup\limits_{(x,y)\in S_j} |D_\alpha u^{(i)}(x,y) - D_\alpha u^{(i')}(x,y)| \le \dfrac{C_3'}{h^{|\alpha|}} (||\ell_i|| + ||\ell_{i'}||)$

Now we claim that

$$(1.25) \qquad ||\ell_i|| \le C_4 h^{m-1} |u|_{m,G_i}$$

To prove this observe that

$$h^{m-1}|u|_{m,G_i} = \left[\int \sum_{\alpha_1+\alpha_2=m} \left(\frac{\partial^m u^{(i)}}{\partial X_i^{\alpha_1} \partial Y_i^{\alpha_2}}\right)^2 dX_i dY_i\right]^{1/2}$$

The quadratic form $[\ ,\ ] = \left[\sum_{\alpha_1+\alpha_2=m}\left(\frac{\partial^m u(i)}{\partial X_i^{\alpha_1}\partial Y_i^{\alpha_2}}\right)^2 dX_i dY_i\right]$

is positive definite on the linear space $\mathfrak{A}_1 \oplus K_i$. Therefore

$\inf_{0\neq z\in\mathfrak{A}_1\oplus K_i} \frac{[z,z]^{1/2}}{||z||} = \lambda_i \neq 0$. λ_i is a function of i and $\mathfrak{h}$. Suppose $\inf_{i,\mathfrak{h}} \lambda_i = 0$.

Then there exists a sequence or cells $\{G_k\}$ such that the corresponding sequence $\lambda_k \to 0$. Now each G_k contains a point (x_k,y_k) and the square $\sigma_k = \{(x,y)\,|\,|x-x_k|\leq h\delta_2, |y-y_k|\leq h\delta_2\}$. Let $[\ ,\ ]'$ denote the bilinear form $[\ ,\ ]$ restricted to the square σ_k. Then there exists a sequence of points (X_k,Y_k), $|X_k|\leq 1-\delta_2$, $|Y_k|\leq 1-\delta_2$ and a corresponding sequence $\{z_k\}$ such that $||z_k||=1$ and $[z_k,z_k]' \to 0$. By compactness, there exists a subsequence $(X_{k_j},Y_{k_j},z_{k_j})$ converging to $(X_{k_o},Y_{k_o},z_{k_o})$ in $\mathbb{R}^2\oplus\mathfrak{A}_1\oplus K_i$ such that $||z_{k_o}|| = 1$ and $[Z_{k_o},Z_{k_o}]' = 0$ which is a contradiction of the positive-definiteness of $[\ ,\ ]'$. This proves the claim. Inequality (1.23) follows. The proof of Inequality (1.24) is similar.

Lemma 2: Let $u\in\mathfrak{F}(\mathfrak{h},p)$. Then

$$(1.26)\qquad |||u|||_{m-1,G_o} \leq \frac{C_9}{h^{\frac{1}{2}}}\,|u|_{m,G_o}$$

Proof: Without loss of generality we may assume that none of the grid lines is parallel to the coordinate axes. Define a function u' on $\mathbb{R}^2$ by letting u' = u on G_o and u' = 0 elsewhere. For any line $y = y_o$ intersecting G_o, pick points (x_a, y_o) and (x_b, y_o) in ∂G such that all points having coordinates (x, y_o) and lying in G be between (x_a, y_o) and (x_b, y_o). For a fixed $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = m-1$, define a function

$$J_\alpha(t) = \begin{cases} D_\alpha u'(x_a + t(x_b - x_a), y_o) & \text{if } (x_a + t(x_b - x_a), y_o) \in G_o \\ 0 & \text{otherwise} \end{cases}$$

Then $J_\alpha(t)$ is piecewise continuous and differentiable over open segments $(-\infty, t_o)$, $(t_o, t_1), \ldots (t_k, \infty)$, where $t_o = o$, $t_k = 1$.

Define $J_\alpha(t_i+) = \lim_{\substack{t \to t_i \\ t - t_i > 0}} J_\alpha(t)$

and $J_\alpha(t_i-) = \lim_{\substack{t \to t_i \\ t - t_i < 0}} J_\alpha(t)$

Let $J'_\alpha(t) = \dfrac{dJ_\alpha}{dt}$ for t in one of the open segments listed above. Then

$J_\alpha(t_{i+1}-) = J_\alpha(t_i+) + \int_{t_i}^{t_{i+1}} J'_\alpha(\tau)d\tau$ for $i=0, 1, \ldots, k-1$. ∴ For any point $t \varepsilon (t_i, t_{i+1})$,

$$J_\alpha(t) = J_\alpha(t_i+) + \int_{t_i}^{t} J'_\alpha(\tau)\, d\tau$$

$$|J_\alpha(t)|^2 = \left[\sum_{j=o}^{i} \{J_\alpha(t_j+) - J_\alpha(t_j-)\} + \int_{t_o}^{t} J'_\alpha(\tau)d\tau\right]^2$$

$$\leqslant 2\left[\sum_{j=o}^{i} \{J_\alpha(t_j+) - J_\alpha(t_j-)\}\right]^2 + 2\left[\int_{t_o}^{t} J'_\alpha(\tau)d\tau\right]^2$$

$$\leqslant 2i\left[\sum_{j=o}^{i} \{J_\alpha(t_j+) - J_\alpha(t_j-)\}^2\right] + 2t\left[\int_{t_o}^{t} |J'_\alpha(\tau)|^2\, d\tau\right]$$

by Cauchy-Schwarz Inequality. From Inequalities (1.23) and (1.24),

$$\sum\{J_\alpha(t_j+) - J_\alpha(t_j-)\}^2 \leqslant 2C_5\sum_k |u|^2_{m,G_k} \leqslant 2C_5|u|^2_{m,G_o}$$

From Inequality (1.25), $|J'_\alpha(\tau)|^2 \leqslant \frac{C_6}{h^2} |u|^2_{m,G_k}$ if

$(x_a+\tau(x_b - x_a), y_o) \varepsilon G_k$ Therefore

$$\int_{t_o}^{t} |J_\alpha'(\tau)|^2 \, d\tau \leq h\sum_k |u|^2_{m,G_k}/h^2 \leq \frac{1}{h} |u|^2_{m,G_o}$$

Condition (e) on the class of admissible grids implies that $i \leq \frac{C_o}{h}$ and

compactness of G_o implies that $t \leq C_7$

Therefore $$|J_\alpha(t)|^2 \leq \frac{C_8^2}{h} |u|^2_{m,G_o}$$

and $$|||D_\alpha u|||_{o,G_o} = |||J_\alpha|||_{o,\mathbb{R}} \leq \frac{C_8}{h^{\frac{1}{2}}} |u|_{m,G_o} \quad \text{for} \quad |\alpha| = m-1$$

By successive integrations in a similar fashion,

$$|||D_\alpha u|||_{o,G_o} \leq \frac{C_8}{h^{\frac{1}{2}}} |u|_{m,G_o} \qquad \text{for} \quad |\alpha| \leq m-1.$$

The lemma follows.

We state without proof the following special form of the Green's formula. The proof is similar to the one given in reference [1]

Lemma 3. There exist linear differential operators $N_k(x,y,D)$ for $(x,y)\varepsilon\bigcup_i\partial G_i$ with coefficients bounded uniformly with respect to all (x,y) and all $\mathfrak{h}$, such that N_k is of order $\leq k$ and for all $u\varepsilon C^{\infty}(\bar{G})$ and $\varphi\varepsilon\mathcal{J}(\mathfrak{h},p)$,

$$(1.28)\qquad \int_{G_o}\varphi Lu = B(\varphi,u)+\sum_{S_k\notin\partial G}\sum_{j=0}^{m-1}\int_{S_k}(D_{n,j}\varphi^{(i_k)} - D_{n,j}\varphi^{(i'_k)})N_{2m-1-j}ud\xi$$

$$+\sum_{S_k\varepsilon\partial G}\sum_{j=0}^{m-1}\int_{S_k}(D_{n,j}\varphi^{(i_k)})N_{2m-1-j}ud\,\xi$$

where the grid line $S_k = \partial G_{i_k}\cap\partial G_{i_k}{}'$ or $\partial G\cap\partial G_{i_k}$

Lemma 4. For a given $u\ \varepsilon\ C^{\infty}(\bar{G})$

$$(1.28)\ \left|\int_{G_o}\phi\, Lu - B(\phi,u)\right| \leqslant C_{13}h^{p+1}|\phi|_{m,G_o}\qquad \text{for all } \phi\ \varepsilon\ \mathcal{J}(\mathfrak{h},p) \text{ and all } \mathfrak{h}$$

Proof: If $\alpha = (\alpha_1, \alpha_2)$, let $\alpha! = \alpha_1!\ \alpha_2!$ and

$(x, y)^{\alpha} = x^{\alpha_1} y^{\alpha_2}$ Then by Taylors' theorem

$$u = \sum_{|\alpha|\leqslant p+m}\frac{1}{\alpha!}\, D^{\alpha}u(x_o, y_o)\,(x - x_o)^{\alpha_1}(y - y_o)^{\alpha_2} + R(x,y)$$

where $|||D^{\alpha} R(x,y)|||_G \leq C_{10} d^{p+m+1-|\alpha|}$ and

d = distance between points (x,y) and (x_o, y_o).

Now any grid line S_k is given by the equations $x = f_k(\xi_k)$, $y = g_k(\xi_k)$.

Let (x_o, y_o) be a point in S_k. Since f_k and g_k are C^{∞} functions and since the number of shapes of grid lines is finite, it is possible to choose C_{10} such that

$$(1.29) \qquad D_{n,j}u = \sum_{r=0}^{p+m-j} a_{n,r}\xi_k^r + R_n(\xi_k)$$

where $|||R_n(\xi_k)|||_{s_k} \leq C_{10}\xi_k^{p+m-j+1}$

Substituting (1.29) in Eq. (1.27)

$$\left|\int_{G_o} \phi L u - B(\phi,u)\right| = \sum_{s_k \notin \partial G}\left[\sum_{j=0}^{m-1}\left\{\sum_{r=0}^{p+j+1-m} b^k_{nj}\left(\phi^{i_k k}_{rj} - \phi^{i'_k k}_{rj}\right) + \int_{s_k}\left(D_{n,j}\phi^{i_k} - D_{n,j}\phi^{i'_k}\right) R'_{k,j}(\xi_k\right.\right.$$

$$+ \sum_{s_k \varepsilon \partial G}\left[\sum_{j=0}^{m-1}\left\{\sum_{r=0}^{p+j+1-m} b^k_{nj}\phi^{i_k k}_{rj} + \int_{s_k} D_{n,j}\phi^{i_k} R'_{k,j}(\xi_k)\right\}\right]$$

when $|||R'_{k,j}(\xi_k)|||_{s_k} \leqslant C_{10}\xi_k^{p+j+2-m}$

Since ϕ satisfies the compatibility conditions (1.8),

$$\phi^{i_k k}_{rj} = \phi^{i'_k k}_{rj} \qquad \text{if} \qquad s_k = \partial G_{i_k} \cap \partial G_{i'_k}$$

$$\phi^{i_k k}_{rj} = o \qquad \text{if} \qquad s_k = \partial G_{i_k} \cap \partial G$$

By Cauchy-Schwarz Inequality,

$$\left\{ \sum_{s_k \notin \partial G} \int_{s_k} (D_{n,j}\phi^{i_k} - D_{n,j}\phi^{i'_k} \right\}^2$$

$$\leq \left[\sum_{s_k \notin \partial G} \int_{s_k} (D_{n,j}\phi^{i_k} - D_{n,j}\phi^{i'_k})^2 \, d\xi_k \right] \left[\sum_{s_k \notin \partial G} \int_{s_k} R'^2_{kj} \, d\xi_k \right]$$

$$\leq C_1 C_{10} \left[h^{2(m-1-j)} \left\{ \sum_{s_k \notin \partial G} \int_{s_k} (|\phi|^2_{m,G_{i_k}} + |\phi|^2_{m,G_{i'_k}}) \right\} \right] \left[\sum_{s_k} \int_{s_k} \xi_k^{2(p+j+2-m)} d\xi_k \right]$$

(using Inequalities (1.23), (1.24))

$$\leq 2C_1 C_{10} \left[h^{2(m-1-j)} \, h|\phi|^2_{m,G_o} \right] \left[h^{2(p+j+2-m)} h \frac{(\text{area of } G)}{\delta^2 h^2} \right]$$

$$\leq C_{11} \; h^{2(p+1)} |\phi|^2_{m,G_o}$$

Similarly $\quad \left\{ \sum_{s_k \in \partial G} \int_{s_k} D_{nj}\phi^{i_k} R'_{kj} \, d\xi_k \right\}^2 \leq C_{11} \, h^{2(p+1)} |\phi|^2_{m,G_o}$

Therefore $\quad \left| \int_{G_o} \phi L u - B(\phi,u) \right| \leq C_{13} \, h^{p+1} \, |\phi|_{m,G_o}$

Theorem 1: Equation (1.21) has a unique solution $\hat{u}\varepsilon\mathcal{F}(h,p)$ and

(1.31) $$|||\hat{u}-\bar{u}|||_{m-1,G_o} \leq C_{26}h^{p+\frac{1}{2}}$$

(1.32) $$|\hat{u}-\bar{u}|_{m,G_o} \leq C_{27}h^{p+1}$$

Proof: Suppose for $\varphi,\hat{u}\varepsilon\mathcal{F}(h,p)$, $B(\varphi,\hat{u}) = 0$ for all φ.

$\therefore \quad 0 = |B(\hat{u},\hat{u})| \geq C|\hat{u}|^2_{m,G_o}$

$\therefore \quad |||\hat{u}|||_{m-1,G_o} = 0$ by Inequality (1.26)

$\therefore \quad \hat{u} = 0$

$\therefore$ Eq. (1.21) has a unique $\hat{u}\varepsilon\mathcal{F}(h,p)$.

Using Taylor's theorem, within each cell, we can write the exact solution as $\bar{u} = {}'u(x_i,y_i) + R(x_i,y_i)$ where ${}'u(x_i,y_i)$ is a polynomial of degree n_i and $||D_\alpha R(x_i,y_i)||_{o,G_i} \leq C_{14}h^{1+n_i-|\alpha|}$ for $|\alpha|\leq n_i$

Let $''U^{ij}_{kl} = \bar{U}^{ij}_{kl} - {}'\bar{U}^{ij}_{kl}$ and $''U^{i,w}_{\alpha} = \bar{U}^{i,w}_{\alpha} - U^{i,w}_{\alpha}$

Then $|''U^{ij}_{kl}| \leq C_{14}h^{n_i+1}$ and $|''U^{i,w}_{\alpha}| \leq C_{14}h^{n_i+1}$

Let $\left\{ {}''U^{ij}_{k\ell}\ ,\ \ U^{i,w}_{\alpha} \right\}_{\text{all } j,w,k,\ell,\alpha} = d\varepsilon V_i$

Let $a = \overline{T}_i^{-1}(d) = (A_{oo}, \ldots, A_{o,n_i})$.

Then $|A_{ij}| \le C_{15}\ ||d|| \le C_{16}\ h^{n_i+1}$

Let $''u(x_i,y_i) = a_{oo} + a_{1o}\ x_i + \ldots\ a_{o,n_i}\ y_i^{n_i}$

where $a_{i,j} = A_{ij}/h^{i+j}$

Then $||D_\alpha\ ''U||_{o,G_i} \le C_{17}\ h^{1+n_i-|\alpha|}$ for $|\alpha| \le n_i$

Let ${}^{o}u = {}'u + {}''u$ and $\widetilde{u} = \overline{u} - {}^{o}u$. Then ${}^{o}u \varepsilon \mathfrak{F}(h,p)$

and $||D_\alpha\ \widetilde{u}\ ||_{o,G_i} \le C_{18}\ h^{1+n_i-|\alpha|}$, $|\alpha| \le n_i$

Let $\Delta u = \hat{u} - {}^{o}u \varepsilon \mathfrak{F}(h,p)$

$$\therefore \quad B(\Delta u,\hat{u}) = \int_{G_o} f\Delta u = \int (\Delta u)\ (L\overline{u})$$

$$= B\ (\Delta u, {}^{o}u+\widetilde{u}) + \left\{ \int (\Delta u)\ (L\overline{u}) - B\ (\Delta u,\overline{u}) \right\}$$

By Inequality (1.29)

$$|B(\Delta u,\Delta u)| \le |B\ (\Delta u,\widetilde{u})| + C_{13}\ h^{p+1}\ \ |\Delta u|_{m,G_o}$$

Now the Cauchy-Schwarz Inequality implies that

(1.33) $|B(\Delta u,\tilde{u})| \leq C_{19} ||\Delta u||_{m,G_o} ||\tilde{u}||_{m,G_o} \leq C_{2o} h^{1+n_i-m} ||\Delta u||_{m,G_o}$

Therefore $|B(\Delta u,\tilde{u})| \leq C_{21}\ h^{(p+1)}\ |\Delta u|_{m,G_o}$ by Inequalities (1.10) and (1.26).

Inequalities (1.2) and (1.33) imply that

$$|\Delta u|_{m,G_o} \leq C_{24}\ h^{(p+1)}$$

Inequality (1.26) implies that

$$|||\Delta u|||_{m-1,G_o} \leq C_{25}\ h^{p+\frac{1}{2}}$$

$$|||\hat{u}-\bar{u}|||_{m-1,G_o} \leq |||\Delta u-\tilde{u}|||_{m-1,G_o} \leq |||\Delta u|||_{m-1,G_o} + |||\tilde{u}|||_{m-1,G_o} \leq C_{26} h^{p+\frac{1}{2}}$$

and $|\hat{u}-\bar{u}|_{m,G_o} \leq C_{27}\ h^{(p+1)}$

REFERENCES

[1] AGMON, S. Lectures on Elliptic Boundary Value Problems. New York. Van Nostrand, 1965.

[2] AHLBERG, J.H., NILSON, E.N., WALSH, J.L. The Theory of Splines and their Applications. New York. Academic Press, 1967.

[3] CIARLET, P.G., SCHULTZ, M.H., VARGA, R.S. Numerical Methods of high-order accuracy for Nonlinear Boundary Value Problems.I. One dimensional problem. Numer.Math.9, 394-430, 1967.

[4] CLOUGH, R.W., TOCHER, J.L. Finite element stiffness matrices for analysis of plate bending. Proceedings of conference on Matrix Methods in Structural Mechanics, Air Force Flight Dynamics Laboratory, Air Force Institute of Technology, W.P. Air Force Base, Dayton, Ohio.p515, December, 1965.

[5] HAMMER, P.C., MARLOWE, O.J., STROUD, A.H. Numerical Integration over Simplexes and Cones. Math. Tables and other Aids to Comp. V. 10, 130 - 136, 1959.