MASSACHUSETTS INSTITUTE OF TECHNOLOGY
ARTIFICIAL INTELLIGENCE LABORATORY

A.I. Memo No. 662 November, 1981

PASSIVE NAVIGATION

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Abstract

A method is proposed for determining the motion of a body relative to a fixed environment using the changing image seen by a camera attached to the body. The optical flow in the image plane is the input, while the instantaneous rotation and translation of the body are the output. If optical flow could be determined precisely, it would only have to be known at a few places to compute the parameters of the motion. In practice, however, the measured optical flow will be somewhat inaccurate. It is therefore advantageous to consider methods which use as much of the available information as possible. We employ a least-squares approach which minimizes some measure of the discrepancy between the measured flow and that predicted from the computed motion parameters. Several different error norms are investigated. In general, our algorithm leads to a system of nonlinear equations from which the motion parameters may be computed numerically. However, in the special cases where the motion of the camera is purely translational or purely rotational, use of the appropriate norm leads to a system of equations from which these parameters can be determined in closed form.

This report describes research done at the Artificial Intelligence Laboratory of the Massachusetts Institute of Technology. Support for the laboratory's artificial intelligence research is provided in part by the Advanced Research Projects Agency of the Department of Defense under Office of Naval Research contract N00014-80-C-0505 and in part by National Science Foundation Grant MCS77-07569.

1. Introduction

In this paper we investigate the problem of passive navigation using optical flow information. Suppose we are viewing a film. We wish to determine the motion of the camera from the sequence of images, assuming that the instantaneous velocity of the brightness patterns, also called the *optical flow*, is known at each point in the image. Several schemata for computing optical flow have been suggested (e.g. [2], [3], [5]). Other papers (e.g. [9], [10], [11]) have previously addressed the problem of passive navigation. Three approaches can be taken towards a solution which we term the *discrete*, the *differential* and the *continuous* approach.

In the discrete approach, information about the movement of brightness patterns at only a *few* points is used to determine the motion of the camera. In particular, using such an approach, one attempts to identify and match discrete points in a sequence of images. Of interest in this case is the photogrammetric problem of determining what the minimum number of points is from which the motion can be calculated for a given number of images [10], [11], [14]. This approach requires that one tracks features, or identifies corresponding features in images taken at different times.

In the differential approach, the first and second spatial partial derivatives of the optical flow are used to compute the motion of a camera [6], [9]. It has been claimed that it is sufficient to know the optical flow and both its first and second derivatives at a single point to uniquely determine the motion [9]. This is incorrect (except for a special case) [1]. Furthermore, noise in the measured optical flow is accentuated by differentiation.

In the continuous approach, the whole optical flow field is used. A major shortcoming of both the local and differential approaches is that neither allows for errors in the optical flow data. This is why we choose the continuous approach and devise a least-squares technique to determine the motion of the camera from the measured optical flow. The proposed algorithm takes the abundance of available data into account and is robust enough to allow numerical implementation.

2. Technical Prerequisites

In this section we review the equations describing the relation between the motion of a camera and the optical flow generated. We use essentially the same notation as [9]. A camera is assumed to move through a static environment. Let a coordinate system X, Y, Z be fixed with respect to the camera, with the Z-axis pointing along the optical axis. If we wish, we can think of the environment as moving in relation to this coordinate system. Any rigid body motion can be resolved into two factors, a translation and a rotation. We will denote by $\vec{T}$ the translational component of the motion of the camera and by $\vec{\omega}$ its angular velocity (see also Figure 1 which is redrawn from [9]). Let the instantaneous coordinates of a point P in the environment be (X, Y, Z).

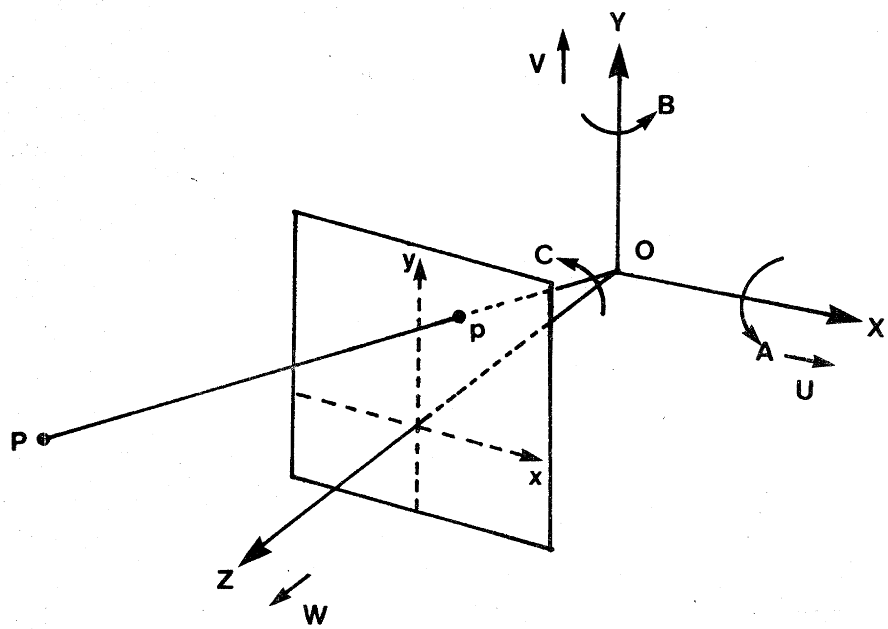

Figure 1. Coordinate Systems

(Note that $Z > 0$ for points in front of the imaging system.) Let $\vec{r}$ be the vector $(X, Y, Z)^T$, where ^{T} denotes the transpose of a vector, then the velocity of P with respect to the X, Y, Z coordinate system, is:

$$\vec{V} = -\vec{T} - \vec{\omega} \times \vec{r}. \tag{1}$$

We define the components of $\vec{T}$ and $\vec{\omega}$ as:

$$\vec{T} = (U, V, W)^T \qquad \vec{\omega} = (A, B, C)^T. \tag{2}$$

Thus we can rewrite (1) in component form:

$$\begin{aligned} X' &= -U - BZ + CY \\ Y' &= -V - CX + AZ \\ Z' &= -W - AY + BX. \end{aligned} \tag{3}$$

where $'$ denotes differentiation with respect to time.

The *optical flow* at each point in the image plane is the instantaneous velocity of the brightness pattern at that point. Let (x, y) denote the coordinates of a point in the image plane (see Figure 1). Since we assume perspective projection between an object point P and the corresponding image point p, the coordinates of p are:

$$x = \frac{X}{Z} \qquad y = \frac{Y}{Z}. \tag{4}$$

The optical flow, denoted by (u, v), at a point (x, y) is:

$$u = x' \qquad v = y'. \tag{5}$$

Differentiating (4) with respect to time and using (3) we obtain the following equations for the optical flow:

$$\begin{aligned} u &= \frac{X'}{Z} - \frac{XZ'}{Z^2} = (-\frac{U}{Z} - B + Cy) - x(-\frac{W}{Z} - Ay + Bx) \\ v &= \frac{Y'}{Z} - \frac{YZ'}{Z^2} = (-\frac{V}{Z} - Cx + A) - y(-\frac{W}{Z} - Ay + Bx). \end{aligned} \tag{6}$$

We can write these equations in the form:

$$u = u_t + u_r \qquad v = v_t + v_r \tag{7}$$

where (u_t, v_t) denotes the translational component of the optical flow and (u_r, v_r) the rotational component:

$$\begin{aligned} u_t &= \frac{-U + xW}{Z} & v_t &= \frac{-V + yW}{Z} \\ u_r &= Axy - B(x^2 + 1) + Cy & v_r &= A(y^2 + 1) - Bxy - Cx. \end{aligned} \tag{8}$$

So far we have considered a single point P. To define the optical flow globally we assume that P lies on a surface defined by a function $Z = Z(X, Y)$ which is positive for all values of X and Y. With any surface and any motion of a camera we can therefore associate a certain optical flow and we say that the surface and the motion *generate* this optical flow.

Optical flow, therefore, depends upon the six parameters of motion of the camera and upon the surface whose images are analyzed. Can all these unknowns be uniquely recaptured solely from optical flow? The answer is no. To see this, consider a surface S_2 which is a dilation by a factor k of a surface S_1. Further, let two motions denoted by M_1 and M_2 have the same rotational component and let their translational components be proportional to each other by the same factor k (we will say that M_1 and M_2 are *similar*). Then the optical flow generated by S_1 and M_1 is the same as the optical flow generated by S_2 and M_2. This follows directly from the definition of optical flow (8). It is still an open question whether there are any other pairs of distinct surfaces and motions which generate the same optical flow.

Determining the motion of a camera from optical flow is much easier if we are told that the motion is purely translational or purely rotational. In the next two sections we will deal with these two special cases. Then we shall analyze the case where no *a priori* assumptions about the motion of the camera are made.

3. Translational Case

In this section we discuss the case where the motion of the camera is assumed to be purely translational. As before, let $\vec{T} = (U, V, W)$ be the velocity of the camera. Then the following equations hold (see (8)):

$$u_t = \frac{-U + xW}{Z} \qquad v_t = \frac{-V + yW}{Z}. \tag{9}$$

3.1. Similar Surfaces and Similar Motions

It will be shown next that if two flows generate the same optical flow, and we know that the motions are purely translational, then the two surfaces are similar and the two camera motions are similar. Let Z_1 and Z_2 be two surfaces and let $\vec{T}_1 = (U_1, V_1, W_1)^T$ and $\vec{T}_2 = (U_2, V_2, W_2)^T$ define two different motions of a camera, such that Z_1 and $\vec{T}_1$ and Z_2 and $\vec{T}_2$ generate the same optical flow:

$$u = \frac{-U_1 + xW_1}{Z_1} \qquad v = \frac{-V_1 + yW_1}{Z_1}, \tag{10}$$

$$u = \frac{-U_2 + xW_2}{Z_2} \qquad v = \frac{-V_2 + yW_2}{Z_2}. \tag{11}$$

Eliminating Z_1, Z_2, u and v from these equations we obtain:

$$\frac{-U_1 + xW_1}{-V_1 + yW_1} = \frac{-U_2 + xW_2}{-V_2 + yW_2}. \tag{12}$$

We can rewrite this equation as:

$$(-U_1 + xW_1)(-V_2 + yW_2) = (-U_2 + xW_2)(-V_1 + yW_1), \tag{13}$$

or:

$$U_1V_2 - xV_2W_1 - yU_1W_2 + xyW_1W_2 = U_2V_1 - xV_1W_2 - yU_2W_1 + xyW_2W_1. \tag{14}$$

Since we assumed that Z_1 and $\vec{T}_1$ and Z_2 and $\vec{T}_2$ generate the same optical flow, the above equation must hold for *all* x and y. Therefore the following equations have to hold:

$$\begin{aligned} U_1V_2 &= U_2V_1 \\ -V_2W_1 &= -V_1W_2 \\ -U_1W_2 &= -U_2W_1. \end{aligned} \tag{15}$$

These equations can be rewritten as:

$$U_1{:}V_1{:}W_1 = U_2{:}V_2{:}W_2 \tag{16}$$

from which it follows that Z_2 is a dilation of Z_1. It is clear that the scaling factor between Z_1 and Z_2 (or equivalently between $\vec{T}_1$ and $\vec{T}_2$) cannot be recovered from the optical flow, regardless of the number of points at which the flow is known. By *uniquely* determining the motion of the camera, we will mean that the motion is uniquely determined up to a constant scaling factor.

3.2. Least-Squares Formulation

In general, the direction of the optical flow at two points in the image plane determine the motion of a camera in pure translation uniquely. There is a drawback however to utilizing so little of the available information. The optical flow we measure is corrupted by noise and it is desirable to develop a robust method which takes this into account. Thus we suggest using a least-squares method [4], [12] to determine the movement parameters and the surface (i.e., the best fit with respect to some norm).

For the following we assume that the image plane is the rectangle $x\epsilon[-w, w]$ and $y[-h, h]$. The same method applies if the image has some other shape. (In fact, it can be used on sub-images corresponding to individual objects in the case that the environment contains objects which may move relative to one another). Furthermore we have to assume that $1/Z$ is a bounded function and that the set of points where $1/Z$ is discontinuous is of measure zero. This condition on $1/Z$ assures us that all necessary integrations can be carried out. We wish to minimize the following expression:

$$\int_{-h}^{h}\int_{-w}^{w}[(u-\frac{-U+xW}{Z})^2+(v-\frac{-V+yW}{Z})^2]dxdy. \tag{17}$$

In this case then, we determine the best fit with respect to the ML_2 norm which is defined as:

$$\| f(x,y) \| = \int_{-h}^{h}\int_{-w}^{w}[f(x,y)]^2 dxdy. \tag{18}$$

The steps in the least-squares method are as follows: First we determine that Z which minimizes the integrand of (17) at every point (x, y). Then we determine the values of U, V and W which minimize the integral (17).

Let us introduce the following abbreviations:

$$\alpha = -U + xW \qquad \beta = -V + yW. \tag{19}$$

Note that the *expected* flow, given U, V and W is simply:

$$\bar{u} = \frac{\alpha}{Z} \qquad \bar{v} = \frac{\beta}{Z}. \tag{20}$$

Then we can rewrite (17) as:

$$\int_{-h}^{h}\int_{-w}^{w}[(u-\frac{\alpha}{Z})^2+(v-\frac{\beta}{Z})^2]dxdy. \tag{21}$$

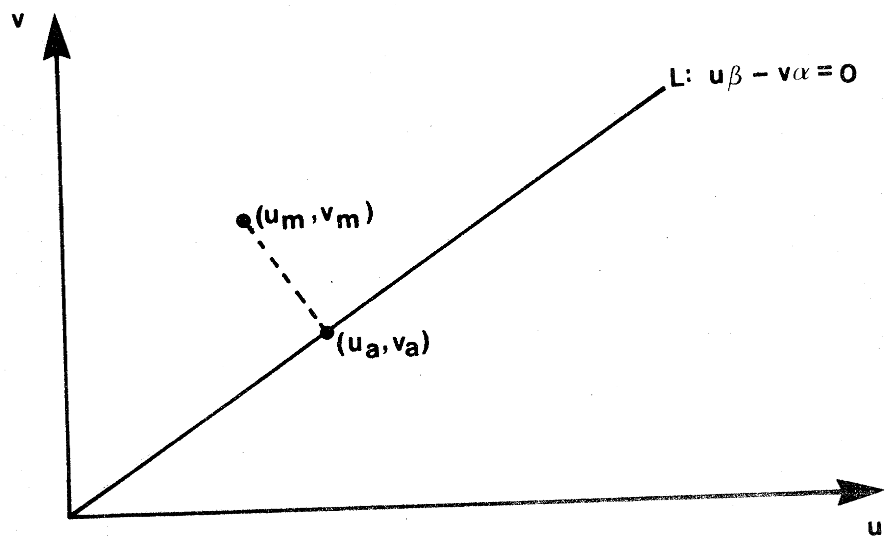

Figure 2. Geometrical Interpretation

We proceed now with the first step of our minimization method. Differentiating the integrand of (17) with respect to Z and setting the resulting expression equal to zero yields:

$$(u - \frac{\alpha}{Z})\frac{\alpha}{Z^2} + (v - \frac{\beta}{Z})\frac{\beta}{Z^2} = 0. \tag{22}$$

Therefore we can write Z as:

$$Z = \frac{\alpha^2 + \beta^2}{u\alpha + v\beta}. \tag{23}$$

This equation, by the way, imposes a constraint on U, V and W, since Z must be positive. We do not make use of this except to help us pick amongst two opposite solutions for the translational velocity later on. Note that now:

$$u - \frac{\alpha}{Z} = \beta\frac{u\beta - v\alpha}{\alpha^2 + \beta^2} \qquad v - \frac{\beta}{Z} = -\alpha\frac{u\beta - v\alpha}{\alpha^2 + \beta^2} \tag{24}$$

and we can therefore rewrite (17) as:

$$\int_{-h}^{h}\int_{-w}^{w} \frac{(u\beta - v\alpha)^2}{\alpha^2 + \beta^2} dxdy. \tag{25}$$

It should be clear, by the way, that uniformly scaling U, V and W does not change the value of (25). This is a reflection of the fact that we can determine the motion parameters only up to a constant factor.

Before proceeding with the second step, we give a geometrical interpretation in Figure 2 of what we have so far. Suppose that the motion parameters U, V, and

W are given. At any given point, say (x_0, y_0), optical flow depends not only upon the motion parameters but also upon the value of Z at that point, Z_0 say. However, the direction of (u, v) does not depend upon Z_0. The point (u, v) must lie along the line L in the uv-plane defined by the equation $u\beta - v\alpha = 0$. Let the *measured* optical flow at (x_0, y_0) be denoted by (u_m, v_m), and let the closest point on the line L be (u_a, v_a). This corresponds to a particular Z_a given by (23). The remaining error is the distance between the point (u_m, v_m) and the line L. The square of this distance is given by the integrand of (25).

For the second step, we differentiate (25) with respect to U, V and W and set the resulting expressions equal to zero:

$$\begin{aligned} \int_{-h}^{h}\int_{-w}^{w} \frac{\beta(u\beta - v\alpha)(u\alpha + v\beta)}{(\alpha^2+\beta^2)^2}\,dxdy &= 0 \\ -\int_{-h}^{h}\int_{-w}^{w} \frac{\alpha(u\beta - v\alpha)(u\alpha + v\beta)}{(\alpha^2+\beta^2)^2}\,dxdy &= 0 \\ \int_{-h}^{h}\int_{-w}^{w} \frac{(y\alpha - x\beta)(u\beta - v\alpha)(u\alpha + v\beta)}{(\alpha^2+\beta^2)^2}\,dxdy &= 0. \end{aligned} \tag{26}$$

Let us introduce the following abbreviation:

$$K = \frac{(u\beta - v\alpha)(u\alpha + v\beta)}{(\alpha^2+\beta^2)^2}. \tag{27}$$

Then equations (26) can be rewritten as:

$$\begin{aligned} \int_{-h}^{h}\int_{-w}^{w} [(-V + yW)K]\,dxdy &= 0 \\ -\int_{-h}^{h}\int_{-w}^{w} [(-U + xW)K]\,dxdy &= 0 \\ \int_{-h}^{h}\int_{-w}^{w} [(-yU + xV)K]\,dxdy &= 0. \end{aligned} \tag{28}$$

The sum of U times the first integral, V times the second integral, and W times the third integral is identically zero. Thus the three equations are linearly dependent. This is to be expected, for if:

$$f(kU, kV, kW) = f(U, V, W), \tag{29}$$

where f is a differentiable function and k a constant, then:

$$U\frac{\partial f}{\partial U} + V\frac{\partial f}{\partial V} + W\frac{\partial f}{\partial W} = 0. \tag{30}$$

The result is also consistent with the fact that only two equations are needed, since the translational velocity can be determined only up to a constant factor. Unfortunately equations (28) are nonlinear in U, V and W and we are not able to show that they have a unique (up to a constant scaling factor) solution.

3.3. Using a Different Norm

There is a way, however, to devise a least-squares method which allows us to display a closed form solution for the motion parameters. Instead of minimizing (17), we will try to minimize the following expression:

$$\int_{-h}^{h}\int_{-w}^{w}[(u-\frac{-U+xW}{Z})^2+(v-\frac{-V+yW}{Z})^2](\alpha^2+\beta^2)dxdy \qquad (31)$$

obtained by multiplying the integrand of (17) by $\alpha^2+\beta^2$. Then we apply the same least-squares method as before to (31). When the measured optical flow is not corrupted by noise, both (31) and (17) can be made equal to zero by substituting the correct motion parameters. We thus obtain the same solution for the motion parameters whether we minimize (31) or (17). If the measured optical flow is not exact, then using expression (31) for our minimization, we obtain the best fit with respect not to the ML_2 norm, but to another norm which we call the $ML_{\alpha\beta}$ norm:

$$\| f(x,y) \|_{\alpha\beta} = \int_{-h}^{h}\int_{-w}^{w}[f(x,y)]^2(\alpha^2+\beta^2)dxdy. \qquad (32)$$

What we have here is a minimization in which the error contributions are *weighted*, greater importance being given to points where the optical flow velocity is larger. This is most appropriate when the measurement of *larger* velocities is more accurate.

Which norm gives the best results depends on the properties of the noise in the measured optical flow. The first norm is better suited to the sitation where the noise in the measurements is *independent* of the magnitude of the optical flow. Note also that if we really want the minimum with respect to the ML_2 norm, we can use the results of the minimization with respect to the $ML_{\alpha\beta}$ norm as starting values in a numerical minimization.

We discuss now our least-squares method in the case where the norm is chosen to be $ML_{\alpha\beta}$. First we determine Z by differentiating the integrand of (31) with respect to Z and setting the result equal to zero. We again get (22):

$$(u-\frac{\alpha}{Z})\frac{\alpha}{Z^2}+(v-\frac{\beta}{Z})\frac{\beta}{Z^2}=0, \qquad (33)$$

from which it follows that (23):

$$Z=\frac{\alpha^2+\beta^2}{u\alpha+v\beta}. \qquad (34)$$

So we want to minimize:

$$\int_{-h}^{h}\int_{-w}^{w}(u\beta - v\alpha)^2 dxdy. \tag{35}$$

Let us call this integral $g(U,V,W)$, then, since:

$$u\beta - v\alpha = (vU - uV) - (xv - yu)W, \tag{36}$$

we have:

$$g(U,V,W) = aU^2 + bV^2 + cW^2 + 2dUV + 2eVW + 2fWU, \tag{37}$$

where:

$$\begin{aligned} a &= \int_{-h}^{h}\int_{-w}^{w} v^2 dxdy & b &= \int_{-h}^{h}\int_{-w}^{w} u^2 dxdy \\ c &= \int_{-h}^{h}\int_{-w}^{w} (xv - yu)^2 dxdy & d &= -\int_{-h}^{h}\int_{-w}^{w} uv dxdy \\ e &= \int_{-h}^{h}\int_{-w}^{w} u(xv - yu) dxdy & f &= -\int_{-h}^{h}\int_{-w}^{w} v(xv - yu) dxdy. \end{aligned} \tag{38}$$

Now $g(U,V,W)$ cannot be negative, and $g(U,V,W) = 0$ for $U = V = W = 0$. Thus a minimum can be found by inspection, but is not what we might have hoped for. In fact, to determine the translational velocity using our least-squares method we have to solve the following homogeneous equation for $\vec{T}$:

$$G\vec{T} = 0 \tag{39}$$

where G is the matrix:

$$G = \begin{pmatrix} a & d & f \\ d & b & e \\ f & e & c \end{pmatrix}. \tag{40}$$

Clearly (39) has a solution other then zero if and only if the determinant of G is zero. Then the three equations (39) are linearly dependent and $\vec{T}$ can be determined up to a constant factor. In general, however, as the data is corrupted by noise, g cannot be made equal to zero for non-zero translational velocity and so $\vec{T} = (0,0,0)^T$ will be the only solution to (39). To see this in another way, note that g has the following form:

$$g(kU,kV,kW) = k^2 g(U,V,W) \tag{41}$$

where k is a constant. Clearly $g(U,V,W)$ assumes its minimal value for $U = V = W = 0$.

What we are really interested in, is determining the *direction* of $\vec{T}$ which minimizes g, for a fixed length of $\vec{T}$. Hence we impose the constraint that $\vec{T}$ be a unit vector.

If $\vec{T}$ is constrained to have unit magnitude, the minimum value of g is the smallest eigenvalue of the matrix G and the value of $\vec{T}$ for which g assumes its minimum can be found by determining the eigenvector corresponding to this eigenvalue [8]. This follows from the observation that g is a quadratic form which can be written as:

$$g(U, V, W) = \vec{T}^T G \vec{T}. \tag{42}$$

Note that G is a positive semidefinite hermitian matrix as $a \geq 0$, $b \geq 0$, $c \geq 0$, $ab \geq d^2$, $bc \geq e^2$ and $ca \geq f^2$. (The last three inequalities follow from the Cauchy-Schwarz inequality [7], [8]). Hence all eigenvalues are real and non-negative and are the solutions λ of the third degree polynomial:

$$\begin{aligned} &\lambda^3 - (a+b+c)\lambda^2 + (ab+bc+ca-d^2-e^2-f^2)\lambda + \\ &(ae^2+bf^2+cd^2-abc-2def) = 0. \end{aligned} \tag{43}$$

There is an explicit formula for the least positive root in terms of the real and imaginary parts of the roots of the quadratic resolvent of the cubic. In our case this gives us the desired smallest root, since the roots cannot be negative. For the sake of completeness, however, various pathological cases that might come up will be discussed next, even though they are of little practical interest.

Note that $\lambda = 0$ is an eigenvalue if and only if G is singular, that is, if the constant term in the polynomial (43) equals zero. In fact, if the determinant of G is zero one can find a velocity $\vec{T}$ which makes g zero. It follows from a theorem in calculus that this happens only when the optical flow is either not corrupted by noise at all or only at a few points. The theorem states that if the integral of the square of a bounded and continuous function is zero then the function itself is zero. Hence errors can only occur at points where the optical flow is discontinuous, and these are exactly the points where the surface defined by Z is discontinuous. (These are also the places where existing methods for computing the optical flow [5] are subject to large errors).

It is impossible for exactly two eigenvalues to be zero, since this would imply that the coefficient of λ in the polynomial (43) equalled zero, while the coefficient of λ^2 did not. That in turn would imply that $ab = d^2$, $bc = e^2$, and $ca = f^2$, while a, b, and c are not all zero. For equality to hold in the Cauchy-Schwarz inequalities, however, u and v must both be proportional to $xv - yu$. This can only be true (for all x and y in the image) if u=v=0. But then all six integrals become zero and consequently all *three* eigenvalues are zero. This situation is of little interest, since it occurs only when the optical flow data is zero everywhere. Then the velocity is zero too. Once the smallest eigenvalue is known, it is straightforward to find the translational velocity which best matches the given data. To determine the eigenvector corresponding to an eigenvalue, say λ_1, we have to solve the following set of linear equations:

$$\begin{aligned}(a-\lambda_1)U+dV+fW&=0\\ dU+(b-\lambda_1)V+eW&=0\\ fU+eV+(c-\lambda_1)W&=0.\end{aligned}\tag{44}$$

As λ_1 is an eigenvalue, equations (44) are linearly dependent. Let us for a moment assume that all eigenvalues are distinct, that is, the rank of the matrix $(G-\lambda I)$, where I is the identity matrix, is two. Then we can use any pair of them to solve for U, V in terms of W say. There are three ways of doing this. For numerical accuracy we may add the three results to get the symmetrical forms:

$$\begin{aligned}U&=(b-\lambda_1)(c-\lambda_1)-f(b-\lambda_1)-d(c-\lambda_1)+e(f+d-e)\\ V&=(c-\lambda_1)(a-\lambda_1)-d(c-\lambda_1)-e(a-\lambda_1)+f(d+e-f)\\ W&=(a-\lambda_1)(b-\lambda_1)-e(a-\lambda_1)-f(b-\lambda_1)+d(e+f-d).\end{aligned}\tag{45}$$

Note that λ_1 will be very small, if the data is good, and one may wish to just approximate the exact solution by using the above equations with λ_1 set to zero. (Then there is no need to find the eigenvalue). In any case, the resulting velocity may now be normalized so that its magnitude equals one. There is one remaining difficulty, arising from the fact that if $\vec{T}$ is a solution to our minimization problem, so is $-\vec{T}$. Only one of these solution will correspond to positive values of Z in equation (34) however. This can be easily seen by evaluating (34) at some point in the image. The case where the two smallest eigenvalues are the same will be discussed in one of the next paragraphs.

There is a simple geometrical interpretation of what we have done so far. To this end we consider the surface defined by $g(U,V,W)=k$ where k is a constant. Note that we can always find a new coordinate system $\tilde{U},\tilde{V},\tilde{W}$ in which $g(U,V,W)$ can be written as:

$$\lambda_1\tilde{U}^2+\lambda_2\tilde{V}^2+\lambda_3\tilde{W}^2=k\tag{46}$$

where λ_i for $i=1,2,3$ are the three eigenvalues of the quadratic form. If the eigenvalues are all non-zero, the surface $g(U,V,W)=k$ is an ellipsoid with three orthogonal semi-axes of length $\sqrt{k/\lambda_i}$. We are particularly interested in the case where the constant k is the smallest eigenvalue. Then all three semi-axes have lengths less than or equal to one. Hence the ellipsoid lies within the unit sphere. If the two smallest eigenvalues are distinct, the unit sphere touches the ellipsoid in two places, corresponding to the largest axis. If the two smaller eigenvalues happen to be the same, however, the unit sphere touches the ellipsoid along a circle and as a result all the velocity vectors lying in a plane spanned by two eigenvectors give equally low errors. Finally, if all three eigenvalues are equal, no direction for $\vec{T}$ is preferred, since the ellipsoid becomes the unit sphere.

The case where exactly one eigenvalue is zero also has a simple geometrical interpretation. The surface defined by $g(U,V,W)=0$ is a straight line, which can be seen easily from the following equation:

$$\lambda_1\tilde{U}^2+\lambda_2\tilde{V}^2=0\tag{47}$$

written for the case when λ_3 is zero. (Remember that λ_1 and λ_2 are both positive.) Clearly the unit sphere intersects this line in exactly two points, one of them corresponding to positive values for Z in equation (34).

The method which we just described can be easily implemented. To this end, the problem can be discretized. An expression similar to (31) can be derived where the integrals are approximated by sums. Our minimization method can then be applied to these sums. The resulting equations are similar to ones described in this section, with summation replacing integration. We implemented the resulting algorithm and tested it using synthetic data including additive noise. Thes results agreed with our expectations.

One can use the ratio of the biggest to the smallest eigenvalue, the so called condition number [13], as a measure of confidence in the computed velocity. The result is very sensitive to errors in the measurements unless this ratio is much bigger than one.

Curiously, the same error integral as (35) is obtained in the case where the ML_{Zuv} norm is used:

$$\| f(x,y) \|_{Zuv} = \int_{-h}^{h}\int_{-w}^{w} [f(x,y)Z(x,y)]^2(u^2+v^2)dxdy. \tag{48}$$

We can arrive at a similar solution by multiplying the integrand in (17) by Z^2 instead of by $\alpha^2+\beta^2$. In that case the minimization is carried out with respect to the ML_Z norm defined by:

$$\| f(x,y) \|_{Z} = \int_{-h}^{h}\int_{-w}^{w} [f(x,y)Z(x,y)]^2 dxdy. \tag{49}$$

Here optical flow velocities for points which are further away are weighted more heavily. This is most appropriate when the measurement of larger velocities is *less* accurate. We end up with a quadratic form similar to g, only the integrals for the six constants corresponding to a, b, c, d, e, and f are a bit more complicated. Curiously they only depend on the direction of the optical flow at each point, not its magnitude.

Also, other constraints could be used. If we insist on $U^2+V^2=1$, for example, we obtain a quadratic instead of a cubic equation, and if we use $W=1$, a linear equation only need to be solved. The disadvantage of these approaches is that the result is sensitive to the orientation of the coordinate axes. Clearly, in the case of exact data, we get the right solution using any of the constraints mentioned above.

3.4. Using a Different Constraint

The minimization scheme discussed in the previous section gives us a unique solution in most cases for the velocity vector $\vec{T}$. Here we propose a slightly different approach which always gives us a unique solution. Note that applying the first step in our minimization method gives us a constraint between the values of Z, the velocity vector and the optical flow at every point. We can in addition assume that $Z = Z_0$ is known at a particular point, say (x_0, y_0). Using the ML_{Zuv} norm in our scheme, we want to minimize:

$$\int_{-h}^{h}\int_{-w}^{w} [uZ-(-U+xW)]^2 + [vZ-(-V+yW)]^2(u^2+v^2)dxdy. \qquad (50)$$

Differentiating (50) with respect to Z, and setting the resulting expression equal to zero, we obtain:

$$Z = \frac{u\alpha + v\beta}{u^2+v^2}. \qquad (51)$$

Thus we propose to solve the minimization scheme under the following constraint:

$$Z_0(u_0^2+v_0^2) - (u_0\alpha + v_0\beta) = 0 \qquad (52)$$

where u_0 and v_0 denote the components of the optical flow measured at (x_0, y_0). The error integral (50) becomes after substituting (51):

$$\int_{-h}^{h}\int_{-w}^{w} (u\beta - v\alpha)^2 dxdy \qquad (53)$$

which is the same as (35) and is denoted by $g(U,V,W)$ (37). Thus we want to minimize:

$$g(U,V,W) + 2\lambda[Z_0(u_0^2+v_0^2) - (u_0\alpha + v_0\beta)] \qquad (54)$$

where λ denotes a Lagrangian multiplier. To determine U, V and W the following linear equations obtained by differentiating (54) with respect to U, V, W and λ have to be solved:

$$\begin{aligned}
&aU + dV + fW + \lambda u_0 = 0\\
&dU + bV + eW + \lambda v_0 = 0\\
&fU + eV + cW - \lambda(x_0u_0 + y_0v_0) = 0\\
&u_0U + v_0V - (x_0u_0 + y_0v_0)W = -Z_0(u_0^2+v_0^2).
\end{aligned} \qquad (55)$$

These equations can be written in the form:

$$G_\lambda \vec{T}_\lambda = \vec{F} \qquad (56)$$

where $\vec{T}_\lambda = (U, V, W, \lambda)^T$ and $\vec{F} = (0, 0, 0, -Z_0(u_0^2+v_0^2))^T$. Let the determinant of G_λ be Δ_0:

$$\begin{aligned}
\Delta_0 =&(d^2-ab)(x_0u_0+y_0v_0)^2 + (e^2-bc)u_0^2 + (f^2-ac)v_0^2 + \\
&2[(de-bf)u_0(x_0u_0+y_0v_0) + (df-ae)v_0(x_0u_0+y_0v_0) + (cd-ef)u_0v_0].
\end{aligned} \qquad (57)$$

Assuming that $\Delta_0 \neq 0$ we can easily determine $\vec{T}_\lambda$ from (55):

$$\vec{T}_\lambda = G_\lambda^{-1}\vec{F}. \tag{58}$$

Introducing the following abbreviation:

$$K = \frac{Z_0(u_0^2 + v_0^2)}{\Delta_0}, \tag{59}$$

we can give these formulae for $\vec{T}_\lambda$:

$$\begin{aligned} U &= K[u_0(bc - e^2) + v_0(ef - cd) + (x_0u_0 + y_0v_0)(bf - de)] \\ V &= K[u_0(ef - cd) + v_0(ac - f^2) + (x_0u_0 + y_0v_0)(ae - df)] \\ W &= K[u_0(de - bf) + v_0(df - ae) + (x_0u_0 + y_0v_0)(d^2 - ab)] \\ \lambda &= K[ae^2 + d^2 + bf^2 - abc - 2def]. \end{aligned} \tag{60}$$

The disadvantage of this approach is that the result depends upon the values of the optical flow at a single point. To circumvent this problem we propose to determine average values for U, V and W in the following manner. First note that we are only interested in the ratios of U/W and V/W which obviously do not depend upon the (unknown) value for Z_0. Equivalently we could determine the value for K from the condition that $\vec{T}$ should have unit length. Hence we can determine values for U, V and W which depend only upon the values of the optical flow at a single point and the coefficient in the matrix G. If we want to remove the dependence of the result on the data at a single point, we can simply average the values obtained for all image points.

In the case where the data is exact, i.e., where the determinant of G, denoted by $\det G$, vanishes, the solution for $\vec{T}$ is the same one as obtained using no constraint in our minimizations scheme. To see this just observe that in that case $\lambda = 0$ as $\lambda = -K\det G$). In the case where $\Delta_0 = 0$, equations (55) have a solution only when $\det G = 0$. We do not have to be concerned with the case where $\Delta_0 = 0$ but $\det G \neq 0$ as we can argue that equations (55) always have to have a solution. Note that our method is based on the condition that Z is a certain function (51) of U, V, W. Hence (52) cannot impose a constraint which would be impossible to satisfy.

4. Rotational Case

Suppose now that the motion of the camera is purely rotational. In order to determine the motion from optical flow we again use a least-squares algorithm with the ML_2 norm described in the previous section. Recall that in this case the optical flow is (see (8)):

$$\begin{aligned} u_r &= Axy - B(x^2 + 1) + Cy \\ v_r &= A(y^2 + 1) - Bxy - Cx. \end{aligned} \tag{61}$$

We will show now in an analogous fashion to section 3.1 that two different rotations, say $\vec{\omega}_1 = (A_1, B_1, C_1)^T$ and $\vec{\omega}_2 = (A_2, B_2, C_2)^T$, cannot generate the same optical flow. Assuming the converse, the following equations have to hold for all values of x and y:

$$\begin{aligned} A_1xy - B_1(x^2+1) + C_1y &= A_2xy - B_2(x^2+1) + C_2y \\ A_1(y^2+1) - B_1xy - C_1x &= A_2(y^2+1) - B_2xy - C_2x \end{aligned} \tag{62}$$

from which we can immediately deduce that $\vec{\omega}_1 = \vec{\omega}_2$.

In general, the direction of the optical flow at two points and its magnitude at one point determine the motion of a camera in pure rotation uniquely. We choose instead to minimize the following expression:

$$\int_{-h}^{h}\int_{-w}^{w} [(u-u_r)^2 + (v-v_r)^2]dxdy. \tag{63}$$

As the motion is purely rotational, the optical flow does not depend upon the distance to the surface and therefore we may omit the first step in our method. Thus we immediately differentiate (63) with respect to A, B and C and set the resulting expressions equal to zero:

$$\begin{aligned} \int_{-h}^{h}\int_{-w}^{w} [(u-u_r)xy + (v-v_r)(y^2+1)]dxdy &= 0 \\ \int_{-h}^{h}\int_{-w}^{w} [(u-u_r)(x^2+1) + (v-v_r)xy]dxdy &= 0 \\ \int_{-h}^{h}\int_{-w}^{w} [(u-u_r)y - (v-v_r)x]dxdy &= 0. \end{aligned} \tag{64}$$

Rewriting the above equations we obtain:

$$\begin{aligned} \int_{-h}^{h}\int_{-w}^{w} [uxy + v(y^2+1)]dxdy &= \int_{-h}^{h}\int_{-w}^{w} [u_rxy + v_r(y^2+1)]dxdy \\ \int_{-h}^{h}\int_{-w}^{w} [u(x^2+1) + vxy]dxdy &= \int_{-h}^{h}\int_{-w}^{w} [u_r(x^2+1) + v_rxy]dxdy \\ \int_{-h}^{h}\int_{-w}^{w} [uy - vx]dxdy &= \int_{-h}^{h}\int_{-w}^{w} [u_ry - v_rx]dxdy \end{aligned} \tag{65}$$

and expanding these equations yields:

$$\begin{aligned} \bar{a}A + \bar{d}B + \bar{f}C &= \bar{k} \\ \bar{d}A + \bar{b}B + \bar{e}C &= \bar{l} \\ \bar{f}A + \bar{e}B + \bar{c}C &= \bar{m}, \end{aligned} \tag{66}$$

where:

$$\bar{a} = \int_{-h}^{h}\int_{-w}^{w}[x^2y^2 + (y^2+1)^2]dxdy$$
$$\bar{b} = \int_{-h}^{h}\int_{-w}^{w}[(x^2+1)^2 + x^2y^2]dxdy$$
$$\bar{c} = \int_{-h}^{h}\int_{-w}^{w}[x^2 + y^2]dxdy \tag{67}$$
$$\bar{d} = -\int_{-h}^{h}\int_{-w}^{w}[xy(x^2+y^2+2)]dxdy$$
$$\bar{e} = -\int_{-h}^{h}\int_{-w}^{w} ydxdy$$
$$\bar{f} = -\int_{-h}^{h}\int_{-w}^{w} xdxdy,$$

and:

$$\bar{k} = \int_{-h}^{h}\int_{-w}^{w}[uxy + v(y^2+1)]dxdy$$
$$\bar{l} = -\int_{-h}^{h}\int_{-w}^{w}[u(x^2+1) + vxy]dxdy \tag{68}$$
$$\bar{m} = \int_{-h}^{h}\int_{-w}^{w}[uy - vx]dxdy.$$

If we call the coefficient matrix in (66) M and the column vector on the right-hand side $\vec{n}$, then we have:

$$M\vec{\omega} = \vec{n}. \tag{69}$$

Thus, provided the matrix M is non-singular, we can compute the rotation as follows:

$$\vec{\omega} = M^{-1}\vec{n}. \tag{70}$$

It is easy to see that the matrix M is non-singular in the special case of a rectangular image plane since then:

$$\bar{a} = 4wh(\frac{h^4}{5} + \frac{2h^2}{3} + 1) + \frac{4w^3h^3}{9}$$
$$\bar{b} = 4wh(\frac{w^4}{5} + \frac{2w^2}{3} + 1) + \frac{4w^3h^3}{9} \tag{71}$$
$$\bar{c} = \frac{4wh}{3}(w^2 + h^2)$$
$$\bar{d} = \bar{e} = \bar{f} = 0.$$

So in the case of a rectangular image plane, the matrix is diagonal, which makes it particularly easy to compute its inverse. In fact, the matrix is diagonal if the image plane is symmetrical about the x-axis and the y-axis. As the extent of the image plane decreases, however, the matrix M becomes ill conditioned. That is inaccuracies in the three integrals ($\bar{k}$, $\bar{l}$, and $\bar{m}$) computed from the observed flow are greatly magnified. This makes sense since we cannot expect to accurately determine the component of rotation about the optical axis when observations are confined to a small cone of directions about the optical axis. Again, in our numerical implementation of the algorithm the integrals in (67) can be approximated by sums.

5. General Motion

We would like now to apply a least-squares algorithm to determine the motion of a camera from optical flow given no *a priori* assumptions about the motion. It is plain that a least-squares method is easiest to use when the resulting equations are *linear* in all the motion parameters. Unfortunately, there exists no norm which will allow us to achieve this goal. We did find a norm, however, which resulted in equations that are linear in *some* of the unknowns and quadratic in the others. We again attack the minimization problem using the $ML_{\alpha\beta}$ norm under the constraint that $U^2 + V^2 + W^2 = 1$. The ensueing equations are polynomials in the unknowns U, V, W, A, B and C and can be solved by a standard iteration method like Newton's method or Bairstow's method [12] or by an interpolation scheme like *regula falsi* [12]. The expression we wish to minimize is:

$$\int_{-h}^{h}\int_{-w}^{w}\{[u-(\frac{\alpha}{Z}+u_r)]^2+[V-(\frac{\beta}{Z}+v_r)]^2\}(\alpha^2+\beta^2)dxdy. \tag{72}$$

The first step is to differentiate the integrand of (72) with respect to Z and set the resulting expression equal to zero:

$$Z=\frac{\alpha^2+\beta^2}{(u-u_r)\alpha+(v-v_r)\beta}. \tag{73}$$

We introduce the Langrangian multiplier λ as before and attempt to minimize:

$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha]^2dxdy+\lambda(U^2+V^2+W^2-1). \tag{74}$$

The equations we have to solve to determine the motion parameters are obtained by differentiation:

$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha][-xy\beta+(y^2+1)\alpha]dxdy=0$$
$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha][(x^2+1)\beta-xy\alpha]dxdy=0$$
$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha][y\beta+x\alpha]dxdy=0 \tag{75}$$
$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha]v_r dxdy+\lambda U=0$$
$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha]u_r dxdy+\lambda V=0$$
$$\int_{-h}^{h}\int_{-w}^{w}[(u-u_r)\beta-(v-v_r)\alpha](-u_r x+v_r y)dxdy+\lambda W=0$$
$$U^2+V^2+W^2=1.$$

Note that the first three of these equations are linear in A, B and C from which these parameters can be determined uniquely in terms of U, V and W. Then we can determine U, V and W from the last four equations by a numerical method. To this end, the problem can be discretized and equations analogous to (75) derived, where summation of the appropriate expressions is used instead of integration.

6. Summary

Our objective was to devise a method for determining the motion of a camera from optical flow which allows for noise in the measured data. The least-squares method which we proposed in this paper meets our goal and is also suitable for numerical implementation. An important application of our results is in passive navigation. Here the path and instantaneous altitude of a vehicle is to be determined from information gleaned about the environment without the emission of sampling radiation from the vehicle.

Acknowledgements: The first author wants to thank Stephen Trilling for helpful comments which improved the style of this paper. We also want to thank Judi Jones for making the drawings.

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