Exponential Speedup of Fixed Parameter Algorithms on $K_{3,3}$-minor-free or K_5-minor-free Graphs*

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Abstract. We present a fixed parameter algorithm that constructively solves the k-dominating set problem on graphs excluding one of the K_5 or $K_{3,3}$ as a minor in time $O(3^{6\sqrt{34k}}n^{O(1)})$. In fact, we present our algorithm for any H-minor-free graph where H is a single-crossing graph (can be drawn on the plane with at most one crossing) and obtain the algorithm for $K_{3,3}(K_5)$-minor-free graphs as a special case. As a consequence, we extend our results to several other problems such as vertex cover, edge dominating set, independent set, clique-transversal set, kernels in digraphs, feedback vertex set and a series of vertex removal problems. Our work generalizes and extends the recent result of exponential speedup in designing fixed-parameter algorithms on planar graphs due to Alber et al. to other (non-planar) classes of graphs.

1 Introduction

According to a survey paper published in year 1998 [HHS98], there are more than 200 research papers published on solving domination-like problems on graphs. Since this problem is very hard and NP-complete even for special kinds of graphs such as planar graphs, much attention has focused on solving this problem on a more restricted class of graphs. It is well known that this problem can be solved on trees [CGH75] or even the generalization of trees, graphs of bounded treewidth [TP93]. The approximability of the dominating set problem has received considerable attention, but it is not known and it is not believed that this problem has constant factor approximation algorithms on general graphs [ACG+99].

Downey and Fellows [DF99] introduced a new concept to handle NP-hardness, namely *fixed parameter tractability*. Unfortunately, according to this theory, it is very unlikely that the k-dominating set problem has a fixed parameter algorithm for general graphs. In contrast, this problem is fixed parameter tractable on planar graphs. The first algorithm for planar k-dominating set was claimed in the book of Downey and Fellows [DF99]. Recently, Alber et al. [ABFN00] demonstrated a solution to the planar k-dominating set in time $O(3^{6\sqrt{34k}}n)$. Indeed, this result was the first non-trivial result for the parameterized version of an NP-hard problem where the exponent of the exponential term grows sublinearly in the parameter. One of the aims of this paper is to generalize this result to non-planar classes of graphs.

A graph G is *H-minor-free* if H cannot be obtained from any subgraph of G by contracting edges. A graph is called a *single-crossing graph* if it can be drawn on the plane with at most

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one crossing. Similar to the approach of Alber et al., we prove that for a single-crossing graph H, the treewidth of any H-minor-free graph G having a k-dominating set is bounded by $O(\sqrt{k})$. We note that planar graphs are both $K_{3,3}$-minor-free and K_5-minor-free, where $K_{3,3}$ and K_5 are both single-crossing graphs. As a result, we generalize current exponential speedup in designing fixed-parameter algorithms on planar graphs to other kinds of graphs and show how we can solve the k-dominating set problem on $K_{3,3}$-minor-free or K_5-minor-free graphs in time $O(3^{6\sqrt{34k}}n^{O(1)})$. The genesis of our results lies in a result of Hajiaghayi et al. [HNRT01] on obtaining the local treewidth of the aforementioned class of graphs.

Using the solution for the k-dominating set problem on planar graphs, Kloks et al. [KC00,CKL01,KLL01] [GKL01] and Alber et al. [ABFN00,AFN01] obtained exponential speedup in solving other problems such as vertex cover, independent set, clique-transversal set, kernels in digraph and feedback vertex set on planar graphs. In this paper also we show how our results can be extended to these problems and many other problems such as variants of dominating set, edge dominating set and a series of vertex removal problems (see Section 6 for details).

This paper is organized as follows. First, we introduce the terminology used throughout the paper, and formally define tree decompositions, treewidth and fixed parameter tractability in Section 2. In Section 3, we introduce the concept of clique-sum graphs, we prove two general theorems concerning the construction of tree decompositions of width $O(\sqrt{k})$ for these graphs and finally we consider the design of fast fixed parameter algorithms for them. In Section 4 we apply our general result on the k-dominating set problem and in Section 5 we describe how this result should be used in order to derive fast fixed parameter algorithms for a series of parameters. In Section 6, we prove some graph theoretic results that provide a framework for designing fixed parameter algorithms for a series of vertex removal problems. In Section 7, we give some further extensions of our results and in Section 8 we end with some conclusions and open problems.

2 Background

2.1 Preliminaries

We assume the reader is familiar with general concepts of graph theory such as (un)directed graphs, trees and planar graphs. The reader is referred to standard references for appropriate background [BM76]. In addition, for exact definitions of various NP-hard graph-theoretic problems in this paper, the reader is referred to Garey and Johnson's book on computers and intractability [GJ79].

Our graph terminology is as follows. All graphs are finite, simple and undirected, unless indicated otherwise. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e in a graph G between u and v by $\{u, v\}$. We define n to be the number of vertices of a graph when it is clear from context. We define the *r-neighborhood* of a set $S \subseteq V(G)$, denoted by $N_G^r(S)$, to be the set of vertices at distance at most r from at least one vertex of $S \subseteq V(G)$; if $S = \{v\}$ we simply use the notation $N_G^r(v)$. The *union* of two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

For generalizations of algorithms on undirected graphs to directed graphs, we consider underlying graphs of directed graphs. The *underlying graph* of a directed graph $H = (V, E)$ is the undirected graph $G = (V, E)$ in which $V(G) = V(H)$ and $\{u, v\} \in E(G)$ if and only if $(u, v) \in E(H)$ or $(v, u) \in E(H)$.

One way of describing classes of graphs is by using *minors*, introduced below.

Definition 1. Contracting *an edge $e = \{u, v\}$ is the operation of replacing both u and v by a single vertex w whose neighbors are all vertices that were neighbors of u or v, except u and v themselves. A graph G is a* minor *of a graph H if H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{C}$ is a* minor-closed *class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is H-*minor-free *if $H \notin \mathcal{C}$.*

For example, a planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors.

2.2 Treewidth

The notion of treewidth was introduced by Robertson and Seymour [RS86] and plays an important role in their fundamental work on graph minors. To define this notion, first we consider the representation of a graph as a tree, which is the basis of our algorithms in this paper.

Definition 2 ([RS86]). *A* tree decomposition *of a graph $G = (V, E)$, denoted by $TD(G)$, is a pair (χ, T) in which $T = (I, F)$ is a tree and $\chi = \{\chi_i | i \in I\}$ is a family of subsets of $V(G)$ such that:*

1. $\bigcup_{i \in I} \chi_i = V$;
2. *for each edge $e = \{u, v\} \in E$ there exists an $i \in I$ such that both u and v belong to χ_i; and*
3. *for all $v \in V$, the set of nodes $\{i \in I | v \in \chi_i\}$ forms a connected subtree of T.*

To distinguish between vertices of the original graph G and vertices of T in $TD(G)$, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The maximum size of a bag in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G ($tw(G)$) is the minimum width over all possible tree decompositions of G.

Many NP-complete problems have linear-time or polynomial-time algorithms when they are restricted to graphs of bounded treewidth. There are a few techniques for obtaining such algorithms. The main technique is called *computing tables of characterizations of partial solutions*. This technique is a general dynamic programming approach, first introduced by Arnborg and Proskurowski [AP89]. Bodlaender [Bod97] described a better presentation of this technique. Other approaches applicable for solving problems on graphs of bounded treewidth are *graph reduction* [ACPS93,BdF96] and *describing the problems in certain types of logic* [ALS88,Cou90].

2.3 Fixed parameter tractability

Developing practical algorithms for NP-hard problems is an important issue. Recently, Downey and Fellows [DF99] introduced a new approach to cope with this NP-hardness, namely *fixed* *parameter tractability.* For many NP-complete problems, the inherent combinatorial explosion is often due to a certain part of a problem, namely a *parameter.* The parameter is often an integer and small in practice. The running times of simple algorithms may be exponential in the parameter but polynomial in the problem size. For example, it has been shown that k-vertex cover has an algorithm with running time $O(kn + 1.271^k)$ [CKJ99] and hence this problem is fixed parameter tractable.

Definition 3 ([DF99]). *A parameterized problem $L \subset \Sigma^* \times \mathbb{N}$ is* fixed parameter tractable (FPT) *if there is an algorithm that correctly decides, for input $(x, k) \in \Sigma^* \times \mathbb{N}$, whether $(x, k) \in L$ in time $f(k)n^c$, where n is the size of the main part of the input x, $|x| = n$, k is a parameter (usually an integer), c is a constant independent of k, and f is an arbitrary function.*

One of the interesting and important properties of fixed parameter tractability is that the definition is *unchanged* if we replace time $f(k)n^c$ by time $f'(k) + n^{c'}$ in the above definition. More precisely:

Lemma 1. *Suppose an algorithm A runs in time $O(f(k)n^c)$ for a parameterized problem (on graphs) with parameter k for some function $f(k)$. We can obtain an algorithm A' for the problem which takes $O(g(k) + n^{c+1})$ time to execute for some function $g(k)$.*

Proof. First we compute a table T containing the solutions for all graphs of size at most $f(k)$. Now if our input graph has $n \leq f(k)$ vertices then we look up the solution in table T, otherwise, i.e. $n > f(k)$, we run algorithm A and obtain the solution in $O(f(k)n^c) = O(n^{c+1})$. □

3 General results on clique-sum graphs

In this section we will define the general framework of our results. A basic tool will be the graph summation operation that will play an important role as it did in the work due to Hajiaghayi et al. [HNRT01,Haj01] to obtain the local treewidth of *clique-sum* graphs, defined formally below.

Definition 4. *Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G'_i ($i = 1, 2$) be obtained from G_i by deleting some (possibly no) edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a* k-sum *G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G'_1 and G'_2 by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the* join set.

In the rest of this section, when we refer to a vertex v of G in G_1 or G_2, we mean the corresponding vertex of v in G_1 or G_2 (or both). It is worth mentioning that $\oplus$ is not a well-defined operator and it can have a set of possible results. The reader is referred to Figure 1 to see an example of a 5-sum operation.

Lemma 2 shows how the treewidth changes when we apply a graph summation operation.

Lemma 2. *For any two graphs G and H,* $\mathrm{tw}(G \oplus H) \leq \max\{\mathrm{tw}(G), \mathrm{tw}(H)\}$.

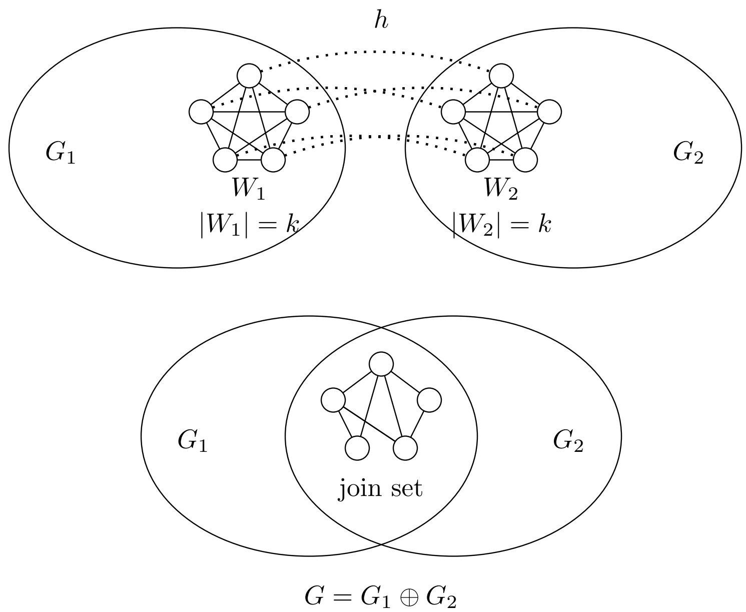

Fig. 1. Example of 5-sum of two graphs

Proof. Let W be the set of vertices of G and H identified during the $\oplus$ operation. Since W is a clique in G, in every tree decomposition of G, there exists a node α such that W is a subset of χ_α [BM93]. Similarly, the same is true for W and a node α' of each tree decomposition of H. Hence, we can construct a tree decomposition of G and a tree decomposition of H and add an edge between α and α'. □

Let s be an integer where $0 \leq s \leq 3$ and $\mathcal{C}$ be a finite set of graphs. We say that a graph class $\mathcal{G}$ is a *clique-sum class* if any of its graphs can be constructed after a sequence of i-sums ($i \leq s$) applied to planar graphs and the graphs in $\mathcal{C}$. We call a graph *clique-sum* if it is a member of a clique-sum class. We call the pair $(\mathcal{C}, s)$ *defining pair* of $\mathcal{G}$ and we call the maximum treewidth of the graphs in $\mathcal{C}$ *base* of $\mathcal{G}$ and *base* of the graphs in $\mathcal{G}$. A series of k-sums (not necessarily unique) which generate a clique-sum graph G are called *a set of clique-sum operations* of G.

According to the result of [RS93], if $\mathcal{G}$ is the class of graphs excluding a single crossing graph (can be drawn on the plane with at most one crossing) H then G is a clique-sum class with defining pair $(\mathcal{C}, s)$ where the base of $\mathcal{G}$ is bounded by a constant c_H depending only to H. In particular, if $H = K_{3,3}$, the defining pair is $(\{K_5\}, 2)$ and $c_H = 4$ [Wag37] and if $H = K_5$ then the defining pair is $(\{V_8\}, 3)$ and $c_H = 4$ [Wag37]. Here by V_8 we mean the graph obtained from a cycle of length eight by joining each pair of diagonally opposite vertices by an edge. For more results on clique-sum classes see [Die89].

From the definition of clique-sum graphs, one can observe that for any clique-sum graph G which excludes a single crossing graph H as a minor, any minor G' of G is also a clique-sum graph which excludes the same graph H as a minor

We call a clique-sum graph class $\mathcal{G}$ α-*recognizable* if there exists an algorithm that for any graph $G \in \mathcal{G}$ outputs in $O(n^\alpha)$ time a sequence of clique sums of graphs of total size $O(|V(G)|)$

that constructs G. We call a graph α*-recognizable* if it belongs in some α-recognizable clique-sum graph class.

Theorem 1 ([KM92]). *The class of K_5-minor-free graphs is a 2-recognizable clique-sum class.*

Asano presented in [Asa85] an $O(n)$ time algorithm for finding a set of clique-sum operations of a graph with no subgraph homomorphic to $K_{3,3}$. As for a cubic graph H (degree of each vertex is at most three), H is a minor of G if and only if G contains a subgraph homeomorphic to H, we have:

Theorem 2 ([Asa85]). *The class of $K_{3,3}$-minor-free graphs is a 1-recognizable clique-sum class.*

A *parameterized graph class* (or just *graph parameter*) is a family $\mathcal{F}$ of classes $\{\mathcal{F}_i, i \geq 0\}$ where $\bigcup_{i\geq 0} \mathcal{F}_i$ is the set of all the graphs and for any $i \geq 0$, $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$. Given two parameterized graph classes $\mathcal{F}^1$ and $\mathcal{F}^2$ and a natural number $\gamma \geq 1$ we say that $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ if for any $i \geq 0$, $\mathcal{F}^1_i \subseteq \mathcal{F}^2_{\gamma\cdot i}$.

In the rest of this paper, we will identify a parameterized problem with the *parameterized graph class* corresponding to its "yes" instances.

Theorem 3. *Let $\mathcal{G}$ be an α-recognizable clique-sum graph class with base c and let $\mathcal{F}$ be a parameterized graph class. In addition, we assume that each graph in $\mathcal{G}$ can be constructed using i-sums where $i \leq s \leq 3$. Suppose also that there exist two positive real numbers β_1, β_2 such that:*

(1) For any $k \geq 0$, planar graphs in $\mathcal{F}_k$ have treewidth at most $\beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be found in linear time.

(2) For any $k \geq 0$ and any $i \leq s$, if $G_1 \oplus_i G_2 \in \mathcal{F}_k$ then $G_1,G_2 \in \mathcal{F}_k$

Then, for any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \max\{\beta_1\sqrt{k} + \beta_2, c\}$ and such a tree decomposition can be constructed in $O(n^\alpha + (\sqrt{k})^s \cdot n)$ time.

Proof. Let $G \in \mathcal{G} \cap \mathcal{F}_k$ and assume that $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ where each G_i, $1 \leq i \leq m$, is either a planar graph or a graph of treewidth at most c. We use induction on m, the number of G_i's. For $m = 1$, $G = G_1$ is either a planar graph that from (1) has treewidth at most $\beta_1\sqrt{k} + \beta_2$ or a graph of treewidth at most c. Thus the basis of the induction is true for both cases. We assume the induction hypothesis is true for $m = h$, and we prove the hypothesis for $m = h + 1$. Let $G' = G_1 \oplus G_2 \oplus \cdots \oplus G_h$ and $G'' = G_{h+1}$. Thus $G = G' \oplus G''$. By (2), both G' and G'' belong in $\mathcal{F}_k$. By the induction hypothesis, $tw(G') \leq \max\{\beta_1\sqrt{k} + \beta_2, c\}$ and from (1) $tw(G'') \leq \max\{\beta_1\sqrt{k} + \beta_2, c\}$. The proof, for $m = h + 1$, follows from this fact and Lemma 2.

To construct a tree decomposition of the aforementioned width, first we construct a tree decomposition of width at most $\beta_1\sqrt{k} + \beta_2$ for each planar graph in linear time. We also note that using Bodlaender's algorithm [Bod96], we can obtain a tree decomposition of width c for any graph of treewidth at most c in linear time (the hidden constant only depends on c). Then having tree decompositions of G_i's, $1 \leq i \leq m$, in the rest of the algorithm, we glue together the tree decompositions of G_i's using the construction given in the proof of Lemma 2. To this end, we introduce an array $Nodes$ indexed by all subsets of $V(G)$ of size at most s. In this array, for each subset whose elements form a clique, we specify a node of the tree decomposition which contains this subset. We note that for each clique C in G_i, there exists a node z of $TD(G)$ such that all vertices of C appear in the bag of z [BM93]. This array is initialized as part of a preprocessing stage of the algorithm. Now, for the $\oplus$ operation between $G_1 \oplus \cdots \oplus G_h$ and G_{h+1} over the join set W, using array $Nodes$, we find a node α in the tree decomposition of $G_1 \oplus \cdots \oplus G_h$ whose bag contains W. Since we have the tree decomposition of G_{h+1}, we can find the node α' of the tree decomposition whose bag contains W by brute force over all subsets of size at most s of bags. Simultaneously, we update array $Nodes$ by subsets of $V(G)$ which form a clique and appear in bags of the tree decomposition of G_{h+1}. Then we add an edge between α and α'. As the number of nodes in a tree decomposition of G_{h+1} is in $O(|V(G_{h+1})|)$ and each bag has size at most $O(\sqrt{k})$ (and thus there are at most $O((\sqrt{k})^s)$ choices for a subset of size at most s), this operation takes $O((\sqrt{k})^s \cdot |V(G_{h+1})|)$ time for G_{h+1}.

The claimed running time follows from the time required to determine a set of clique-sum operations, the time required to construct tree decompositions, the time needed for gluing tree decompositions together and the fact that $\sum_{i=1}^{m} |V(G_i)| = O(|V(G)|)$. □

Theorem 4. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}$ be some parameterized graph class. Suppose also for some positive real numbers $\alpha, \beta_1, \beta_2, \delta$ the following hold:*

(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be decided and constructed (if it exists) in $O(n^\alpha)$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^\alpha)$ time.

(2) Given a tree decomposition of width at most w of a graph, there exists an algorithm deciding whether the graph belongs in $\mathcal{F}_k$ in $O(\delta^w n)$ time.

Then there exists an algorithm deciding in $O(\delta^{\beta_1\sqrt{k}+\beta_2} + n^\alpha)$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}_k$.

Proof. First, we can test membership in $\mathcal{G}$ in $O(n^\alpha)$ time. Then we can apply the algorithm from (1) and (assuming success) supply the resulting tree decomposition to the algorithm from (2).

4 Fixed parameter algorithms for dominating set

In this section, we will describe some of the consequences of Theorems 3 and 4 on the design of efficient fixed parameter algorithms for a series of parameterized problems where their inputs are clique-sum graphs.

A *dominating set* of a graph G is a set of vertices of G such that each of the rest of vertices has at least one neighbor in the set. We represent the *k-dominating set* problem with the parameterized graph class $\mathcal{DS}$ where $\mathcal{DS}_k$ contains graphs which have a dominating set of size $\leq k$. Our target is to show how we can solve the k-dominating set problem on clique-sum graphs, where H is a single-crossing graph, in time $O(c^{\sqrt{k}} n^{O(1)})$ instead of the current algorithms which run in time $O(c^k n^{O(1)})$ for some constant c. By this result, we extend the current exponential speedup in designing algorithms for planar graphs [AFN01] to a more generalized class of graphs. In fact, planar graphs are both $K_{3,3}$-minor-free and K_5-minor-free graphs, where both $K_{3,3}$ and K_5 are single-crossing graphs.

The following result implies that condition (1) of Theorem 3 is satisfied for $\mathcal{DS}$ for $\beta_1 = 6\sqrt{34}$ and $\beta_2 = 8$.

Theorem 5 ([ABFN00]). *Suppose we have a planar graph G and a dominating set of size at most k. Then the treewidth of G is at most $6\sqrt{34}\sqrt{k}+8$ and one can obtain a tree decomposition of this width in linear time.*

The next lemma shows that condition (2) of Theorem 3 is also correct.

Lemma 3. *If $G = G_1 \oplus_m G_2$ has a k-dominating set, then both G_1 and G_2 have dominating sets of size at most k.*

Proof. Let the k-dominating set of G be S and W be the join set of $G_1 \oplus_k G_2$. W.l.o.g. we show that G_1 has a dominating set of size k. If $S_1 = S \cap V(G_1)$ is a dominating set for G_1 then the result immediately follows, otherwise there exists vertex $w \in V(G_1)$ which is dominated by a vertex $v \in V(G_2) - V(G_1)$. One can observe that all such vertex w are in W. Since $v \in S$, but $v \notin S_1$, set $S'_1 = S_1 + \{w\}$ has at most k vertices and since W is a clique in G_1, S'_1 is a dominating set of size at most k in G_1. □

We can now apply Theorem 3 for $\beta_1 = 6\sqrt{34}$ and $\beta_2 = \max\{8, c\}$.

Theorem 6. *If $\mathcal{G}$ is an α-recognizable clique-sum class of base c, then any member G of $\mathcal{G}$ where its dominating set has size at most k has treewidth at most $6\sqrt{34}\sqrt{k} + \max\{8, c\}$ and the corresponding tree decomposition of G can be constructed in $O(n^\alpha)$ time.*

Theorem 6 tells us that condition (1) of Theorem 4 is satisfied. The following result shows that for the graph parameter $\mathcal{DS}$ condition (2) of Theorem 4 is also satisfied for $\delta = 3$.

Theorem 7 ([ABFN00]). *If a tree decomposition of width w of a graph is known, then a minimum dominating set can be determined in time $O(3^w \cdot n)$, where n is the number of vertices.*

The following theorem is a direct consequence of Theorems 4, 6 and 7.

Theorem 8. *There is an algorithm that in $O(3^{6\sqrt{34}\sqrt{k}}n + n^\alpha)$ time solves the k-dominating set problem for any α-recognizable clique-sum graph.*

Corollary 1. *There is an algorithm that in $O(3^{6\sqrt{34}\sqrt{k}}n)$ ($O(3^{6\sqrt{34}\sqrt{k}}n + n^2)$) time solves the k-dominating set problem for $K_{3,3}$(K_5)-minor-free graphs.*

5 Algorithms for parameters bounded by dominating set number

We provide a general methodology for deriving fast fixed parameter algorithms in this section. First, we consider the following theorem which is an immediate consequence of Theorem 4.

Theorem 9. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}^1$, $\mathcal{F}^2$ be two parameterized graph classes where $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ for some natural number $\gamma \geq 1$. Suppose also that there exist positive real numbers $\alpha, \beta_1, \beta_2, \delta$ such that:*

(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}^2_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be decided and constructed (if it exists) in $O(n^\alpha)$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^\alpha)$ time.

(2) There exists an algorithm deciding whether a graph of treewidth $\leq w$ belongs in $\mathcal{F}^1_k$ in $O(\delta^w n)$ time.

Then

(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}^1_k$ all have treewidth at most $\beta_1\sqrt{\gamma k} + \beta_2$ and such a tree decomposition can be constructed in $O(n^\alpha)$ time.

(2) There exists an algorithm deciding in $O(\delta^{\beta_1\sqrt{\gamma k}+\beta_2} + n^\alpha)$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}^1_k$.

Proof. Consequence (1) follows immediately from the definition of $\preccurlyeq_\gamma$. Consequence (2) follows from Theorem 4.

The idea of our general technique is given by the following theorem that is a direct consequence of Theorems 6 and 9.

Theorem 10. *Let $\mathcal{F}$ be a parameterized graph class satisfying the following two properties:*

(1) It is possible to check membership in $\mathcal{F}_k$ of a graph G of treewidth at most w in $O(\delta^w n)$ time for some positive real number δ.

(2) $\mathcal{F} \preccurlyeq_\gamma \mathcal{DS}$.

Then

(1) Any clique sum graph G of base c in $\mathcal{F}_k$ has treewidth at most $\max\{6\sqrt{34}\sqrt{\gamma k} + 8, c\}$.

(2) We can check whether an input graph G is in $\mathcal{F}_k$ in $O(\delta^{6\sqrt{34}\sqrt{\gamma k}} n + n^\alpha)$ [1] on an α-recognizable clique-sum graph of base c.

In what follows we will explain how Theorem 10 applies for a series of graph parameters. In particular, we will explain why Conditions (1) and (2) are satisfied for each problem.

[1] In the rest of this paper, we assume that constants, e.g. c, are small and they do not appear in the powers, since they are absorbed into the O notation.

5.1 Variants of the dominating set problem

A *k-dominating set with property Π* on an undirected graph G is a k-dominating set D of G which has the additional property Π and the *k-dominating set with property Π problem* is the task to decide, given a graph $G = (V, E)$, a property Π, and a positive integer k, whether or not there is a k-dominating set with property Π. Some examples of this type of problems, which are mentioned in [ABFN00,TP93,TP97], are the *k-independent dominating set* problem, the *k-total dominating set* problem, the *k-perfect dominating set* problem, the *k-perfect independent dominating set* problem also known as *k-perfect code* and the *k-total perfect dominating set* problem. For each Π, we will denote the corresponding dominating set problem as $\mathcal{DS}^{\Pi}$.

Another variant is *the weighted dominating set problem* in which we have a graph $G = (V, E)$ together with an integer weight function $w : V \to \mathbb{N}$ with $w(v) > 0$ for all $v \in V$. The weight of a vertex set $D \subseteq V$ is defined as $w(D) = \sum_{v \in D} w(v)$. A k-weighted dominating set D of an undirected graph G is a dominating set D of G with $w(D) \leq k$. The *k-weighted dominating set* problem is the task of deciding whether or not there exits a k-weighted dominating set. We will use the parameterized class $\mathcal{WDS}$ to denote the k-weighted dominating set problem.

Condition (1) of Theorem 10 holds for $\delta = 4$ because of the following.

Theorem 11 ([ABFN00]). *If a tree decomposition of width w of a graph is known, then a solution to $\mathcal{DS}^{\Pi}$ or to $\mathcal{WDS}$ can be determined in at most $O(4^w \cdot n)$ time.*

Clearly, $\mathcal{DS}^{\Pi} \preccurlyeq_1 \mathcal{DS}$ and $\mathcal{WDS} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) also holds. Therefore Theorem 10 holds for $\gamma = 1$ and $\delta = 4$ for $\mathcal{DS}^{\Pi}$ and $\mathcal{WDS}$.

Another related problem is the Y-domination problem ($\mathcal{DS}^{Y}$) introduced in [BBHS96].

Definition 5. *Let Y be a finite set of integers. A* Y-domination *is an assignment $f : V \to Y$ such that for each vertex x, $f(N[x]) = \sum_{v \in N[x]} f(x) \geq 1$ where $N[x]$ stands for the neighborhood of x including x itself. An* efficient Y-domination *is an assignment f with $f(N[x]) = 1$ for all vertices $x \in V$. The* value *of a Y-domination f is $|\{x|f(x) > 0\}|$. The weight of a Y-domination is $\sum_{x \in V} f(x)$. Two Y-dominations are* equivalent *if they have the same closed neighborhood sum at every vertex. The* Y-domination problem *asks whether the input graph G has an efficient Y-domination of value at most k.*

Using the generalized dynamic programming approach, Cai and Kloks [KC00] presents an algorithm which runs in time $O(|Y|^w n)$ to decide whether a graph G of treewidth at most w has an efficient Y-domination of value at most k. It is worth mentioning that, according to Bange et al. [BBHS96], a graph G has an efficient Y-domination if and only if all equivalent Y-dominations have the same weight, and thus there is no need to worry about the actual weight of an efficient Y-domination. Therefore, we have that Condition (1) of Theorem 10 holds for $\delta = |Y|$.

One can easily see that for Y-domination f of a graph $G = (V, E)$, $D = \{x|f(x) > 0\}$ is a dominating set, since each vertex x has at least one vertex with a positive number assigned to it in $N[x]$. Thus if f is a Y-domination of G with value at most k, then G also has a dominating set of size k. Therefore, $\mathcal{DS}^{Y} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) holds as well. Theorem 10 applies for $\gamma = 1$ and $\delta = |Y|$.

5.2 Vertex cover

The k-vertex cover problem ($\mathcal{VC}$) asks whether there exists a subset C of at most k vertices such that every edge of G has at least one endpoint in C. This problem is one of the most popular problems in combinatorial optimization.

A great number of researchers believe that there is no polynomial time approximation algorithm achieving an approximation factor strictly smaller than $2 - \epsilon$, for a positive constant ϵ, unless $P = NP$. Currently, the best known lower bound for this factor is 1.36 [DS] and the best upper bound is 2 which can be obtained easily. The best current fixed-parameter tractable algorithm has time $O(1.271^k + k|V|)$ [CKJ99]. In this section, we present an exponentially faster algorithm for this problem on clique-sum graphs.

Without loss of generality, we can restrict our attention to graphs with no vertex of degree zero. One can observe that if a graph G has a vertex cover of size k, then it has also a k-dominating set. Therefore $\mathcal{VC} \preccurlyeq_1 \mathcal{DS}$ and condition (1) of Theorem 10 holds. Moreover, Condition (2) holds because we can solve the vertex cover problem in time $O(2^w)$ if we know the tree decomposition of width w of a graph G [AFN01]. Therefore, Theorem 10 applies for $\gamma = 1$ and $\delta = 2$ for the k-vertex problem.

A simple standard reduction to the problem kernel due to Buss and Goldsmith [BG93] is as follows: Each vertex of degree greater than k must be in the vertex cover of size k, since otherwise, not all edges can be covered. Thus we can obtain a subgraph G' of G which has at most k^2 edges and at most $k^2 + k$ vertices and k' is obtained from k reduced by the number of vertices of degree more than k. Chen et al. [CKJ99] showed that in Buss and Goldsmith's approach one can even obtain a problem kernel with at most $2k$ vertices in $O(nk + k^3)$ time. Thus, using this result with the consequence (2) of Theorem 10 for $\mathcal{VC}$, we obtain the following result.

Theorem 12. *We can find a k-vertex cover in time $O(2^{6\sqrt{34}\sqrt{k}}k + kn + n^\alpha)$ on an α-recognizable clique-sum graph.*

5.3 Edge dominating set

Another related problem is the edge dominating set problem $\mathcal{EDS}$ that given a graph G asks for a set $E' \subseteq E$ of k or fewer edges such that every edge in E shares at least one endpoint with some edge in E'. Again without loss of generality we can assume that graph G has no vertex of degree zero.

One can observe that if a graph G has a k-edge dominating set E', we can obtain a vertex cover of size $2k$ by including both end-points of each edge $e \in E'$. This means that $\mathcal{EDS} \preccurlyeq_2 \mathcal{VC}$. In the previous section we show that $\mathcal{VC} \preccurlyeq_1 \mathcal{DS}$ therefore, the Condition (2) of Theorem 10 holds for $\mathcal{EDS}$ when $\gamma = 2$. Condition (1) holds because the edge dominating set problem can be solved in $c_{\mathsf{eds}}^w n$ [Bod88,Bak94] (c_{eds} is a small constant) on a tree decomposition of width w for a graph G. We conclude that Theorem 10 applies for $\gamma = 2$ and $\delta = c_{\mathsf{eds}}$.

Theorem 13. *We can find a k-edge dominating set in time $O({c_{eds}}^{6\sqrt{34}\sqrt{2k}}n + n^\alpha)$ on an α-recognizable clique-sum graph.*

5.4 Clique-transversal set

A *clique-transversal set* of a connected graph G is a subset of vertices intersecting all the maximal cliques of G [BNR96,CCCY96,AST91,GR00]. Since the vertex cover problem is NP-complete even restricted to triangle-free planar graphs [CKL01,Ueh96], the clique-transversal problem remains NP-complete on clique-sum graphs. The k*-clique transversal* problem $\mathcal{CT}$ asks whether the input graph has a clique-transversal set of size $\leq k$.

If a graph G has a k-clique-transversal, then it has a dominating set of size at most k, since every vertex of G is contained in at least one maximal clique. This implies that $\mathcal{CT} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) of Theorem 10 holds for $\gamma = 1$. Using the general dynamic programming technique, we can solve the k-clique-transversal problem on a graph G of treewidth at most w in $O(c_{\mathsf{ct}}^{w} n)$ for some constant c_{ct} (the approach is very similar to Chang et al. [CKL01]). Therefore, Theorem 10 applies for $\gamma = 1$ and $\delta = c_{\mathsf{ct}}$.

Theorem 14. *We can find a k-clique-transversal set in time $O({c_{ct}}^{6\sqrt{34}\sqrt{k}} n + n^{\alpha})$ on an α-recognizable clique-sum graph.*

5.5 Minimum Maximal Matching

A *matching* in a graph G is a set E' of edges without common endpoints. A matching in G is *maximal* if there is no other matching in G containing it. The k-maximal matching problem $\mathcal{MM}$ asks whether an input graph G has a maximal matching of size $\leq k$.

Let E' be the edges of a maximal matching of G. Notice that the set of endpoints of the edges in E' is a dominating set of G. Therefore $\mathcal{MM} \preccurlyeq_2 \mathcal{DS}$ and the Condition (2) of Theorem 10 holds. Condition (1) holds because the problem can be solved in $c_{\mathsf{mm}}^{w} n$ [Bod88] on a tree decomposition of width w for a graph G. Hence Theorem 10 gives the following result.

Theorem 15.
(1) Any clique-sum graph of base c with a minimum maximal marching of size k has treewidth $\leq 6\sqrt{34}\sqrt{2k} + \max\{8, c\}$.
(2) One can decide whether an α-recognizable clique-sum graph G has a minimum maximal matching of size at most k in time $O(c_{\mathsf{mm}}^{6\sqrt{34}\sqrt{2k}} n + n^{\alpha})$.

5.6 Kernels in digraphs

A set S of vertices in a digraph $D = (V, A)$ is a kernel if S is independent and every vertex in $V - S$ has an out-neighbor in S. It has been shown that the problem of deciding whether a digraph has a kernel is NP-complete [GJ79]. Franenkel [Fra81] showed that the kernel problem remains NP-complete even for planar digraphs D with indegree and outdegree at most 2 and total degree at most 3. The k-kernel problem $\mathcal{KER}$ asks whether a graph has a kernel of size k. Moreover, we define the co-$\mathcal{KER}$ problem as the one asking whether an n-vertex graph has a kernel of size $n - k$.

Here, we again observe that if a digraph D has a kernel of size at most k, then its underlying graph G has a dominating set of cardinality at most k. Also for a connected digraph $D = (V, A)$

and kernel K, $V-K$ is a dominating set in the underlying graph of D. Resuming these two facts we have $\mathcal{KER} \preccurlyeq_1 \mathcal{DS}$ and co-$\mathcal{KER} \preccurlyeq_1 \mathcal{DS}$ and Condition (2) of Theorem 10 folds for both problems. We note that a slight variation of Condition (1) also holds because Guting [GKL01] gives a $O(3^w kn)$ time algorithm solving the k-kernel problem on graphs of treewidth at most w using the general dynamic programming approach. The straightforward adaptation of Theorem 10 to this variation of Condition (1) gives the following.

Theorem 16.
(1) Any clique-sum graph of base c that has a kernel of size k or $n-k$ *has treewidth* $\leq 6\sqrt{34}\sqrt{k}+\max\{8,c\}$.
(2) One can decide whether an α*-recognizable clique-sum graph G of base c has a kernel of size k in time* $O(3^{6\sqrt{34}\sqrt{k}}nk+n^{\alpha})$ *for some constant c.*
(3) One can decide whether an α*-recognizable clique-sum graph G of base c has a kernel of size* $n-k$ *in time* $O(3^{6\sqrt{34}\sqrt{k}}n(n-k)+n^{\alpha})$ *for some constant c.*

5.7 Independent set

Here we present a fixed parameter algorithm for the *maximum independent set* problem ($\mathcal{MIS}$) in which one asks for a subset $V' \subseteq V$ of maximum size such that no two vertices in V' are joined by an edge in E. In the *k-maximum independent set* problem, we want to know whether a graph G has a maximum independent set of size at most k. We can observe that if a clique-sum graph G has a maximum independent set of size at most k, then each of the planar graph in its clique-sum operations has a maximum independent set of size at most k. According to the paper due to Alber et al. [AFN01], if the maximum independent set of a planar graph G is at most k then its treewidth is at most $4\sqrt{6k}$ and we can obtain a tree decomposition of this width in linear time. This makes it possible to apply Theorem 3 and derive Condition (1) of Theorem 4. As it is possible to solve the k-maximum independent set problem for graphs of treewidth at most w in $O(2^w n)$ time [AFN01] Condition (2) also holds and we have the following.

Theorem 17.
(1) Any clique-sum graph of base c with maximum independent set of size at most k has treewidth at most $6\sqrt{34}\sqrt{k}+\max\{8,c\}$.
(2) There is an algorithm that decides whether an α*-recognizable clique-sum graph of base c has a maximum independent set of size at most k in time* $O(2^{6\sqrt{34}\sqrt{k}}n+n^{\alpha})$.

Also, we note that if a graph G has an $(n-k)$-independent set then it has a vertex cover of size at most k and visa versa. Using Theorem 12 we have:

Theorem 18. *One can decide whether an* α*-recognizable clique-sum graph G has a* $n-k$*-independent set in time* $O(2^{6\sqrt{34}\sqrt{k}}k+kn+n^{\alpha})$.

6 Fixed parameter algorithms for vertex removal problems

In this section, we focus our attention to $K_{3,3}$-minor-free graphs and K_5-minor-free graphs and present some general results allowing the construction of $O(c^{\sqrt{k}}n)$ time algorithms for a series of vertex removal problems.

We start with some definitions. For any graph class $\mathcal{G}$ and any non-negative integer k the graph class k-almost*($\mathcal{G}$)* contains any graph $G = (V, E)$ where there exists a subset $S \subseteq V(G)$ of size at most k such that $G[V - S] \in \mathcal{G}$. We note that using this notation if $\mathcal{G}$ contains all the edgeless graphs or forests then k-almost($\mathcal{G}$) is the class of graphs with vertex cover $\leq k$ or feedback vertex set $\leq k$.

A useful notion is that of a *strong k-cut*, defined below. A graph $G = (V, E)$ has a *k-cut* $S \subseteq V$ when $G[V - S]$ is disconnected and $|S| = k$. Let G_1, G_2 be two of the connected components of $G[V - S]$. Given a component $G_1 = (V_1, E_1)$ of $G[V - S]$ we define its *augmentation* as the graph $G[V_1 \cup S]$ in which we add all edges among vertices of S. We say a k-cut S *minimally separates* G_1 and G_2 if each vertex of S has a neighbor in G_1 and G_2. A graph $G = (V, E)$ has a *strong k-cut* $S \subseteq V$ if $|S| = k$ and $G[V - S]$ has at least k connected components and each pair of them is minimally separated by S.

We say that G is the result of the multiple k-clique sum of $G_1, \ldots, G_r$ with respect to some join set W if $G = G_1 \oplus_k \cdots \oplus_k G_r$ where the join set is always W and such that W is a strong k-cut of G.

Lemma 4. *Let k be a positive integer and let G be a graph with a strong k-cut S where $1 \leq k$. Then the treewidth of G is bounded above by the maximum of the treewidth of each of the augmented components of G after removing S.*

Proof. The proof is very similar to the proof of Lemma 2 and hence omitted. □

Lemma 5. *Let $G = (V, E)$ be a graph with a strong k-cut S where $1 \leq k \leq 3$. Then if G belongs to some minor-closed graph class $\mathcal{G}$ then any of the augmented components of G after removing S is also k-connected and belongs to $\mathcal{G}$.*

Proof. Let $G_1, \ldots, G_r$ be the pairwise minimally separated components of $G[V - S]$ $(r \geq k)$. W.l.o.g. we will prove that if $\tilde{G}_1$ is the augmentation of G_1 then $\tilde{G}_1 \in \mathcal{G}$. Consider the graph $G' = G[S \cup \bigcup_{1 \leq i \leq k} V(G_i)]$ and then contract in G' all the edges in $\bigcup_{2 \leq i \leq k} G_i$. That way, each of $G_i, 2 \leq i \leq r$ collapses to a single vertex v_i connected to all the vertices in S. It is now easy to see that if we contract any perfect matching of the edges between $\{v_2, \ldots, v_r\}$ and S we obtain $\tilde{G}_1$. □

We now need the following adaptation of the results of [KM92] and [Asa85] (Theorems 1 and 2).

Lemma 6. *Let G be a connected $K_{3,3}$-free graph and let $\mathcal{S}$ be the set of its strong i-cuts, $1 \leq i \leq 2$. Then G can be constructed after a sequence of multiple i-clique sums, $1 \leq i \leq 2$, applied to planar graphs or K_5's where each of these multiple sums has a member of $\mathcal{S}$ as join set. Moreover this sequence can be constructed by an algorithm in $O(n)$ time.*

Lemma 7. *Let G be a connected K_5-free graph and let $\mathcal{S}$ be the set of its strong i-cuts, $1 \leq i \leq 3$. Then G can be constructed after a sequence of multiple i-clique sums, $1 \leq i \leq 3$, applied to planar graphs or V_8's where each of these multiple sums has a member of $\mathcal{S}$ as join set. Moreover this sequence can be constructed by an algorithm in $O(n^2)$ time.*

For each K_5-free or $K_{3,3}$-free graph G, we say that its *valence* is the number of its strong i-cuts, $i \geq 1$. In the rest of this section, we will assume that $\alpha = 2$ when we refer to K_5-free graphs and that $\alpha = 1$ when we refer to $K_{3,3}$-free graphs. We now demonstrate the following variance of Theorem 3.

Theorem 19. *Let $\mathcal{G}$ be a $K_{3,3}(K_5)$-minor-free graph class and let $\mathcal{F}$ be any minor-closed parameterized graph class. Suppose that there exist real numbers $\beta_0 \geq 4, \beta_1$ such that any planar graph in $\mathcal{F}_k$ has treewidth at most $\beta_1\sqrt{k}+\beta_0$ and such a tree decomposition can be found in linear time. Then graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_0$ and such a tree decomposition can be constructed in $O(n)$ $(O(n^2))$ time.*

Proof. Let $G \in \mathcal{G} \cap \mathcal{F}$ be such a graph. The proof is based on an induction on its valence. If the valence of G is 0 the result is clear. Suppose that the result is correct for any graph of valence $< r$. We will prove that it also holds for graphs with valence $r \geq 1$. Let W be a strong 3-cut of G. From Lemma 5 all the augmented components of $G - W$ are $K_{3,3}(K_5)$-minor-free graphs in $\mathcal{F}_k$. Applying the induction hypothesis, we have that each of the augmented components has treewidth at most $\beta_1\sqrt{k} + \beta_0$. Using now lemma 4, we derive the same bound for G. The construction is very similar to the construction mentioned in Theorem 3 and hence omitted. □

We define $\mathcal{T}_r$ to be the class of graphs with treewidth $\leq r$. It is known that for $1 \leq i \leq 2$, $\mathcal{T}_i$ is exactly the class of K_{i+2}-minor-free graphs (see e.g. [Bod98]). We now present a series of consequences of Theorem 19 for solving a series of vertex deletion problems on $K_{3,3}$-minor-free graphs and K_5-minor-free graphs. First, we need the following combinatorial lemma.

Lemma 8. *Planar graphs in k-*almost*$(\mathcal{T}_2)$ have treewidth $\leq 6\sqrt{34}\sqrt{k} + 8$. Moreover, such a tree decomposition can be found in linear time.*

Proof. Our target is to prove that planar graphs in k-almost$(\mathcal{T}_2)$ are subgraphs of planar graphs in $\mathcal{DS}_k$ and the result will be a consequence of Theorem 5.

Let G be a planar graph and S be a set of $\leq k$ vertices in G where $G[V-S]$ is K_4-minor-free. From Lemmas 4 and 5 we can assume that G does not have strong 1- or 2-cuts. A consequence of this is that all the vertices of G have degree at least 3. Another consequence is that two faces of G can have in common either a vertex or an edge (otherwise, a strong 2-cut appears). Consider any planar embedding of G. We call a face of this embedding *exterior* if it contains a vertex of S, otherwise we call it interior. For each exterior face choose a vertex in S and connect it with the rest of its vertices. We call the resulting graph H and we note that (a) G is a subgraph of H, (b) $H[V-S] = G[V-S]$, and (c) all the vertices of the exterior faces of H are dominated by some vertex in S. We claim that S is a dominating set of H. Suppose, towards a contradiction, that there is a vertex v that is not dominated by S. From (c) we can assume that all of the faces containing v are interior. Let H' be the graph induced by the vertices of these faces. As they are all interior, H' should be a subgraph of $H[V-S]$. Let $(x_1,\ldots,x_q,x_1)$ be a cyclic order of the neighbors of v and notice that $q \geq 3$. Let also F_i be the face of H containing the vertices $x_i, v, x_{\mathsf{next}(i)}$, $1 \leq i \leq q$, where $\mathsf{next}(i) = (i+1) \mod q+1$. We note that all these faces are pairwise distinct otherwise v will be a 1-cut for H and G. Let P_i be the path connecting x_i and $x_{\mathsf{next}(i)}$ in H' avoiding v and containing only vertices of F_i. Recall now that two faces of H have either v or an edge containing v in common. Therefore, it is impossible two paths P_i, $P_j, i \neq j$, share an internal vertex. This implies that the contraction of all the edges but one of each of these paths transforms H' to a wheel W_q that, as $q \geq 3$, can be further contracted to a K_4 (a weel W_q is the graph constructed taking a cycle of length q and conecting all its vertices with a new vertex v). As H' is a subgraph of the graph $H[V-S]$ (b) implies that $G[V-S]$ contains a K_4, and this is a contradiction. As now S is a dominating set for H the treewidth of H is at most $6\sqrt{34}\sqrt{k}+8$. From (a) we have that G is a subgraph of a planar graph in $\mathcal{DS}_k$ and this completes the proof of the theorem. □

We conclude the following general result.

Theorem 20. *Let $\mathcal{G}$ be any class of graphs with treewidth ≤ 2. Then any $K_{3,3}$(K_5)-minor-free graph in k-almost($\mathcal{G}$) has treewidth $\leq 6\sqrt{34}\sqrt{k}+8$. Moreover, such a tree decomposition can be found in linear time.*

Proof. Let G be such a graph and let $S \subseteq V(G)$ such that $G[V-S] \in \mathcal{G}$. We note that treewidth$(G[V-S]) \leq 2$. Therefore, $G \in k$-almost$(\mathcal{T}_2)$ and the result follows from Theorem 19 and Lemma 8. □

Combining Theorems 4 and 20 we conclude the following.

Theorem 21. *Let $\mathcal{G}$ be any class of graphs with treewidth ≤ 2. Suppose also that there exists an $O(\delta^w n)$ algorithm that decides whether a given graph belongs in k-almost($\mathcal{G}$) for graphs of treewidth at most w. Then, one can decide whether a $K_{3,3}$(K_5)-minor-free graph belongs in k-almost($\mathcal{G}$) in time $O(\delta^{6\sqrt{34}\sqrt{k}}n + n^{\alpha})$.*

If $\{O_1,\ldots,O_r\}$ is a finite set of graphs, we denote as minor-excl$(O_1,\ldots,O_r)$ the class of graphs that are O_i-minor-free for $i=1,\ldots,r$.

As examples of problems for which Theorems 20 and 21 can be applied, we mention the problems of checking whether a graph, after removing k vertices, is *edgeless* ($\mathcal{G} = \mathcal{T}_0$), or has *maximum degree* ≤ 2 ($\mathcal{G}$ = minor-excl$(K_{1,3})$), or becomes a *a star forest* ($\mathcal{G}$ = minor-excl(K_3, P_3)), or a *caterpillar* ($\mathcal{G}$ = minor-excl$(K_3$, subdivision of $K_{1,3})$), or a *forest* ($\mathcal{G} = \mathcal{T}_1$), or *outerplanar* ($\mathcal{G}$ = minor-excl$(K_4, K_{2,3})$), or *series-parallel*, or has treewidth $\leq k$ ($\mathcal{G} = \mathcal{T}_2$).

We consider the cases where $\mathcal{G} = \mathcal{T}_0$ and $\mathcal{G} = \mathcal{T}_1$ in the next two subsections.

6.1 Feedback vertex set

A *feedback vertex set (FVS)* of a graph G is a set U of vertices such that every cycle of G passes through at least one vertex of U. The previous known fixed parameter algorithms for solving the k-feedback vertex set problem has running time $O((2k+1)^k n^2)$ [DF99] and alternatively time $O((917k^4)!(n+e))$ [Bod92]. Also there exists a randomized algorithm which needs $O(c4^k kn)$ time with probability at least $1-(1-\frac{1}{4^k})^{c4^k}$ [BBYG00]. The *k-feedback vertex set* problem ($\mathcal{FVS}$) asks whether an input graph has a feedback vertex set of size $\leq k$.

Kloks et al. [KLL01] proved that the feedback vertex set problem on planar graphs of treewidth at most w can be solved in $O(c_{\mathsf{fvs}}^w n)$ time for some constant c_{fbs}. The complexity of their algorithm is based on the fact that the number of edges of a planar graph is bounded by a simple linear funcion of its vertices (i.e. $3n-6$). As we have similar bound $3n-5$ for $K_{3,3}(K_5)$-minor-free graphs [Asa85,KM92], we can easily prove that the algorithm of [KLL01] works also for the more general case. Therefore, Theorem 21 can be applied for $\mathcal{G}=\mathcal{T}_1$ and $\delta=c_{\mathsf{fvs}}$ and we have the following.

Theorem 22. *For any $K_{3,3}(K_5)$-minor-free graph G the following hold.*
(1) If G has a feedback vertex set of size at most k then G has treewidth at most $6\sqrt{34}\sqrt{k}+8$.
(2) We can check whether G has a feedback vertex set of size $\leq k$ in $O(c_{\mathsf{fvs}}^{6\sqrt{34}\sqrt{k}}n+n^{\alpha})$ time.

Theorem 22 generalizes the results of [KLL01] to the non-planar $K_{3,3}(K_5)$-minor-free graphs.

6.2 Improving bounds for vertex cover

Alber et al. [AFN01] proved that planar graphs in $\mathcal{VC}_k$ have treewidth at most $4\sqrt{3}\sqrt{k}+5$. Applying Theorem 20, we have that Condition (1) of Theorem 4 holds for $K_{3,3}(K_5)$-minor-free graphs when $\beta_1=4\sqrt{3}$ and $\beta_2=5$. Also, as we mentioned in Subsection 5.2 it is possible to decide in $O(2^w n)$ time if a graph has a vertex cover of size at most k. Therefore, Condition (2) holds for $\delta=2$. Concluding, we have the following improvement of the results of Subsection 5.2 for $K_{3,3}(K_5)$-minor-free graphs.

Theorem 23. *For any $K_{3,3}(K_5)$-minor-free graph G the following hold.*
(1) If G has a vertex cover of size at most k then G has treewidth at most $4\sqrt{3}\sqrt{k}+5$.
(2) We can check whether G has a vertex cover of size $\leq k$ in $O(2^{4\sqrt{3}\sqrt{k}}n+n^{\alpha})$ time.
(3) We can check whether G has a vertex cover of size $\leq k$ in $O(2^{4\sqrt{3}\sqrt{k}}k+kn+n^{\alpha})$ time.

Since $\mathcal{EDS} \preccurlyeq_2 \mathcal{VC}$, we can also obtain an $O({c_{\mathsf{eds}}}^{4\sqrt{3}\sqrt{2k}}n+n^{\alpha})$-time algorithm for the edge dominating set problem.

7 Further extensions

In this section, we obtain fixed parameter algorithms with exponential speedup for k-vertex cover and k-edge dominating set on graphs more general than $K_{3,3}(K_5)$-minor-free graphs. Our approach, similar to the Alber's et al. approach [AFN01], is a general one that can be applied to other problems.

Baker [Bak94] developed several approximation algorithms to solve NP-complete problems for planar graphs. To extend these algorithms to other graph families, Eppstein [Epp00] introduced the notion of bounded local treewidth, defined formally below, which is a generalization of the notion of treewidth. Intuitively, a graph has bounded local treewidth (or locally bounded treewidth) if the treewidth of an r-neighborhood of each vertex $v \in V(G)$ is a function of r, $r \in \mathbb{N}$, and not $|V(G)|$.

Definition 6. *The* local treewidth *of a graph G is the function $\mathrm{ltw}^G : \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximum treewidth of an r-neighborhood in G. We set $\mathrm{ltw}^G(r) = \max_{v \in V(G)}\{\mathrm{tw}(G[N_G^r(v)])\}$, and we say that a graph class $\mathcal{C}$ has* bounded local treewidth (or locally bounded treewidth) *when there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $r \in \mathbb{N}$, $\mathrm{ltw}^G(r) \leq f(r)$.*

A graph is called an *apex graph* if deleting one vertex produces a planar graph. Eppstein [Epp00] showed that a minor-closed graph class $\mathcal{E}$ has bounded local treewidth if and only if $\mathcal{E}$ is H-minor-free for some apex graph H.

So far, the only graph classes studied with small local treewidth are the class of planar graphs [Epp00] and the class of clique-sum graphs, which includes minor-free graphs like $K_{3,3}$-minor-free or K_5-minor-free graphs [HNRT01]. It has been proved that for any planar graph G, $\mathrm{ltw}^G(k) \leq 3k-1$ [HNRT01], and for any $K_{3,3}$-minor-free or K_5-minor-free graph G, $\mathrm{ltw}^G(k) \leq 3k+4$ [Epp00]. For these classes of graphs, there are efficient algorithms for constructing tree decompositions.

Eppstein [Epp00] showed how the concept of the kth outer face in planar graphs can be replaced by the concept of the kth layer (or level) in graphs of locally bounded treewidth. The kth layer (L_k) of a graph G consists of all vertices at distance k from an arbitrary fixed vertex v of $V(G)$. We denote *consecutive layers from i to j* by $L[i,j] = \cup_{i \leq k \leq j} L_k$.

Here we generalize the concept of *layerwise separation*, introduced in Alber's et al. work [AFN01] for planar graphs, to general graphs.

Definition 7. *Let G be a graph layered from a vertex v, and r be the number of layers. A* layerwise separation of width w and size s *for G is a sequence $(S_1, S_2, \cdots, S_r)$ of subsets of V, with property that $S_i \subseteq \bigcup_{j=i}^{i+(w-1)} L_j$; S_i separates layers L_{i-1} and L_{i+w}; and $\sum_{j=1}^{r} |S_j| \leq s$.*

Here we relate the concept of layerwise separation to parameterized problems.

Definition 8. *A parameterized problem P has* layerwise Separation Property (LSP) *of width w and size-factor d, if for each instance (G,k) to problem P, graph G admits a layerwise separation of width w and size dk.*

For example, we can obtain constants $w = 2$ and $d = 2$ for the vertex cover problem. In fact, consider a k-vertex cover C on a graph G and set $S_i = (L_i \cup L_{i+1}) \cap C$. Sets S_i's form a layerwise separation. Similarly, we can get constants $w = 2$ and $d = 2$ for the edge dominating set problem.

Lemma 9. *Let P be a parameterized problem on instance (G, k) that admits a problem kernel of size dk. Then the parameterized problem P on the problem kernel has LSP of width 1 and size-factor d.*

Proof. Consider the problem kernel (G', k') for an instance (G, k) and obtain layering L' for G' from arbitrary vertex v. Then clearly the sequence $S_i = L'_i$ for $i = 1, \cdots, r'$ (r' is the number of layers), is a layerwise separation of width 1 and size $k' \leq dk$ for G'. □

In fact, using Lemma 9 and the problem kernel of size $2k$ (see Subsection 5.2) for the vertex cover problem, this problem has the LSP of width 1 and size-factor 2.

The proof of the following Theorem is very similar to the proof of Theorem 12 of Alber's et al. work [AFN01] and hence omitted.

Theorem 24. *Suppose for a graph G, $ltw(G) \leq cr + d$ and a tree decomposition of width $ch + d$ can be constructed in $O(n^\alpha)$ for any h consecutive layers (h is a constant). Also assume G admits a layerwise separation of width w and size dk. Then we have $tw(G) \leq 2\sqrt{6dk} + cw + d$. Such a tree decomposition can be computed in time $O(n^\alpha)$.*

Now, since for any H-minor-free graph G, where H is a single-crossing graph, $ltw(G) \leq 3r + c_H$ and $tw(L[i, j]) \leq 3(j - i + 1) + c_H$ [HNRT01], we have the following.

Corollary 2. *For any H-minor-free graph G, where H is a single-crossing graph, that admits a layerwise separation of width w and size dk, we have $tw(G) \leq 2\sqrt{6dk} + 3w + c_H$.*

Since we can construct the aforementioned kind of tree decompositions for $K_{3,3}(K_5)$-minor-free graphs in $O(n)(O(n^2))$ and their local treewidth is $3r + 4$ [HNRT01], the following result follows immediately.

Corollary 3. *For any $K_{3,3}(K_5)$-minor-free graph G, that admits a layerwise separation of width w and size dk, we have $tw(G) \leq 2\sqrt{6dk} + 3w + 4$. Such a tree decomposition can be computed in time $O(n)$ $(O(n^2))$.*

For example, using Corollary 3, we know that if a $K_{3,3}(K_5)$-minor-free graph G has a k-vertex cover then $tw(G) \leq 4\sqrt{3k} + 10$. Since we can solve the vertex cover problem in time $O(2^w)$ if we know the tree decomposition of width w of a graph G, we have the following.

Theorem 25. *We can find a k-vertex cover in time $O(2^{4\sqrt{3k}}n)$ $(O(2^{4\sqrt{3k}}n + n^2))$ on a $K_{3,3}(K_5)$-minor-free graph G.*

Similarly, we have the following.

Theorem 26. *One can find a k-edge dominating set in time $O({c_{\mathsf{eds}}}^{4\sqrt{3k}}n)(O({c_{\mathsf{eds}}}^{4\sqrt{3k}}n + n^2))$ on a $K_{3,3}(K_5)$-minor-free graph G for some constant c_{eds}, if it exists.*

In fact, we have this general theorem.

Theorem 27. *Suppose for a graph G , $ltw(G) \leq cr + d$ and a tree decomposition of width $ch + d$ can be constructed in time $O(n^\alpha)$ for any h consecutive layers. Let P be a parameterized problem on G such that P has the LSP of width w and size-factor d and there exists an $O(\delta^w n)$-time algorithm, given a tree decomposition of width w for G, decides whether problem P has a solution of size k on G.*

Then there exists an algorithm which decides whether P has a solution of size k on G in time $O(\delta^{2\sqrt{6dk}+cw+d} n + n^\alpha)$.

8 Conclusions and future work

In this paper, we considered H-minor-free graphs, where H is a single-crossing graph, and proved that if these graphs have a k-dominating set then their treewidth is at most $c\sqrt{k}$ for some small constant c. As a consequence, we obtained exponential speedup in designing FPT algorithms for several NP-hard problems on these graphs, especially $K_{3,3}$-minor-free or K_5-minor-free graphs. In fact, our approach is a general one that can be applied to several problems which can be reduced to the dominating set problem as discussed in Section 5 or to problems that themselves can be solved exponentially faster on planar graphs [AFN01]. Here, we present several open problems that are possible extensions of this paper.

One topic of interest is finding other problems for which the technique of this paper can be applied. Moreover, introducing other classes of graphs than H-minor-free graphs, where H is a single-crossing graph, on which the problems can be solved exponentially faster for parameter k is an interesting question.

For several problems in this paper, Kloks et al. [CKL01,KLL01,GKL01,KC00] introduced a reduction to the problem kernel on planar graphs. Since $K_{3,3}$-minor-free graphs and K_5-minor-free graphs are very similar to planar graphs in the sense of having a linear number of edges and not having a clique of size six, we believe that one might obtain similar results for these graphs. Working in this area was beyond the scope of this paper, but still it would be instructive.

As mentioned before, Lemma 20 holds for any class of graphs with treewidth ≤ 2. It is an open problem whether it is possible to generalize it for any class of graphs of treewidth $\leq h$ for arbitrary fixed h. Moreover, there exists no general method for designing $O(\delta^w n)$ time algorithms for vertex removal problems in graphs with treewidth $\leq w$. If this becomes possible, then Theorem 21 will have considerable algorithmic applications.

Finally, as a matter of practical importance, obtaining a better constant coefficient than $6\sqrt{34}$ for the treewidth of planar graphs having a k-dominating set, which has a direct improvement to our results, would be quite interesting.

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