UNIVERSITY OF ILLINOIS
DIGITAL COMPUTER

AUXILIARY
LIBRARY ROUTINE E 11 - 311

TITLE: Evaluation of Exponentially Weighted Infinite Integrals by Quadrature (Hermite Quadrature)

TYPE: Closed subroutine, with one program parameter

NUMBER OF WORDS: 18 + 2N (see below)

DURATION: N(1.8 + T) milliseconds, where T is the duration in milliseconds of the auxiliary subroutine.

TEMPORARY STORAGE: Location 0 (may be used by auxiliary subroutine)

ENTRY: When this routine is located at y, entry is made by the orders

p	-- aF
	50 pF
p + 1	26 yF

,

where a is the location of the auxiliary subroutine which computes the values of the function to be integrated. When control is returned to the right side of p + 1, the computed integral will be in the accumulator register and location y + 14.

DESCRIPTION: To evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx,$$

this routine uses a form of Gaussian Quadrature appropriate to the interval $(-\infty, \infty)$ and the weighting function e^{-x^2}:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \approx \frac{1}{2^P} \sum_{k=1}^{N} A_k f(x_k). \qquad (1)$$

The values A_k and x_k are chosen in a manner such as to give no truncation error when f(x) is a polynomial of degree 2N - 1 or less. In the case where the factor e^{-x^2} does not occur explicitly in the integrand,

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2}[e^{x^2} g(r)]\,dx \approx \frac{1}{2^Q}\sum_{k=1}^{N} A_k\, e^{(x_k)^2}$$

$$g(x_k) = \frac{1}{2^Q} \sum_{k=1}^{N} B_k\, g(x_k). \qquad (2)$$

It is assumed that the function e^{x^2} $g(x)$ may be closely approximated by a polynomial function.

Because the actual values of the points x_k and the weights A_k and B_k may exceed 1, they have been scaled down by powers of two. P and Q are defined in equations (1) and (2), and R is defined below in equation (3).

N	R	P(for A_k)	Q(for B_k)
1	0	1	1
2	0	0	1
3	1	1	1
4	1	0	1
5	2	0	1
6	2	0	1
7	2	0	1
8	2	0	1
9	2	0	1
10	2	0	1
11	2	0	1
12	2	0	0
13	3	0	0
14	3	0	0
15	3	0	0
16	3	0 (see note)	0
17	3	0 "	0
18	3	0 "	0
19	3	0 '	0
20	3	0 "	0

[Note: Because the weights for these values of N become quite small, there may be appreciable roundoff errors for certain integrands].

The auxiliary subroutine which computes $f(x_k)$ <u>must</u> take the scaling of these values of x_k into account. The function values computed by the auxiliary are assumed to lie in the range $-1 \leq f(x_k) < 1$.

The closed auxiliary subroutine is entered from the main routine with x_k^* in the accumulator and link in Q; control is returned to the main routine with $f(x_k)$ in the accumulator.

USE: To use this routine the programmer copies the integration routine first on his program tape, and <u>immediately after</u> it the parameters, points x_k, and weights A_k or B_k appropriate to his needs. These latter numbers appear on the tail of the library tape, labeled by the number N of points at which the function is to be evaluated, and the type of weights (A_k or B_k) to be used.

SCALING: The scaling of the values of x_k is such that the auxiliary subroutine is presented with x_k^*, where

$$x_k^* = 2^{-R} x_k, \; 0 \leq R \leq 3, \qquad (3)$$

and the largest x_k^* satisfied $1/2 \leq (x_k^*)_{max} < 1$ for all N. The computed integral is scaled down by 2^P (or 2^Q). For the convenience of the programmer, the above scale factors are contained in the subroutine parameter at location y + 16, in the following form:

$$(y+16) = 0R\ (y+18)\ 0P\ (y+18+N), \qquad \text{for A weights}$$
$$= 0R\ (y+18)\ 0Q\ (y+18+N) \qquad \text{for B weights.}$$

ACCURACY: The truncation error due to omission of powers of x higher than 2N is

$$\frac{n!\sqrt{\pi}}{2^n (2n)!} f^{(2n)}(z), \quad \text{where z is some point in } (-\infty, \infty).$$

There will also be round-off errors which may be significant; for a discussion of these see the write-up of library routine E 5 - 195.

REFERENCES: National Bureau of Standards Journal of Research, Vol. 48, p. 111 (1952).

G. Szegö, Orthogonal Polynomials.

F. G. Tricomi, Vorlesungen über Orthogonalreithen

DATE October 12, 1960

PROGRAMMED BY John Ehrman

APPROVED BY J. Snyder

nj

LOCATION	ORDER	NOTES PAGE 1 E 11
	00 K	
0	L5 16L	Preset point and weight addresses
	46 5L	
1	42 7L	
	K5 F	
2	46 6L	Plant subroutine entry and link
	42 13L	
3	41 14L	clear sum box
	2S 4L	
4	S5 F	Save L.S.P. of sum
	40 15L	
5	L5 ( )F	get x_k^*; link to Q
	50 5L	
6	26 ( )F	jump to auxiliary
	40 F	
7	L5 15L	L.S.P. to A, weight in Q
	50 ( )F	
8	74 F	x $f(x_k)$, accumulate M.S.P.
	L4 14L	
9	40 14L	
	L5 8L	
10	L4 5L	step x_k address
	46 5L	
11	F5 7L	step A_k address
	42 7L	
12	L0 17L	
	36 4L	
13	L5 14L	exit via link
	22 ( )F	
14	00 F	temporary store for M.S.P.
	00 F	
15	00 F	" " " L.S.P.
	00 F	
16	0R 18L	
	0P [18+N]L	Coefficient addresses and scale constants

LOCATION	ORDER	NOTES PAGE 2 E11
17	75 15L	end constant
	50 [18+2N]L	
18	x_1^*	
.	.	
.	.	
.	.	
17+N	x_N^*	
18+N	A_1 or B_1	
.	.	
.	.	
.	.	
17+2N	A_N or B_N	