UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE V 6 - 202

TITLE Associated Legendre Functions

TYPE Closed

NUMBER OF WORDS 80 (71 plus the 9 of Program R 1 which is attached.)

TEMPORARY STORAGE 0-10 inclusive and $(k_n + 2)(k_n + 1)$ words at S 3

PRESET PARAMETER S 3 contains the address at which the table of functions $K_0^0(x)$, $K_1^0(x)$, $K_1^1(x)$, ... $K_m^n(x)$ is to be placed

DURATION $[25 + 24k + 12k^2]$ms. Approximately (this expression holds if $k \geq 2$)

ACCURACY The maximum errors that can occur due to round off are:

$$\epsilon_k^m \text{ (= error in } K_k^m) = m2^{-39} + \sum_{j=0}^{m} \binom{m}{j} x^j \epsilon_{k-m+j}^0$$

where

$$\epsilon_k^0 \text{ (= error in } K_k^0) = 2^{-39} \sum_{i=0}^{k-2} (2x)^i$$

(Note that $\epsilon_0^0, \epsilon_1^0 \leq 2^{-40}$)

In general the errors are reasonably less than these maxima. See Discussion of Error below.

DESCRIPTION AND ENTRY Enter with $-.985 \leq x \leq -2^{-19}$, $2^{-19} \leq x \leq .99$ in the accumulator and the program parameter:

p	50	k_nF	
	50	pF	$0 < k_n \leq 12$
p + 1	26	qF	

where k_n is the maximum order of the functions desired. q is the address of this routine. $K_k^m(x)$ will be placed in the location [1/2 (k + 1) (k) + m] S3, i,e. K_0^0 in S3, K_1^0 in 1S3, K_1^1 in 2S3, K_2^0 in 3S3, K_2^1 in 4S3, K_2^2 in 5S3, ..., $K_{k_n}^{k_n}$ in [1/2 (k_n + 2)(k_n+ 1)] S3 If $k_n \geq 13$ the functions K_k^m will exceed scale. If k_n is 0, K_0^0, K_1^0, and K_1^1 will be computed.

NOTE This program contains the library routine R1- 116 Square Root as an appended part. It is hence available for use in other connections and would not need to be placed a second time in the machine. It can be entered in the usual fashion at word 71L of this routine.

DISCUSSION OF METHOD The functions K_k^m (x) computed by this routine are defined as

$$K_k^m (x) = \frac{1}{2\sqrt{\pi}} \bar{P}_k^m (x) = \sqrt{\frac{2k+1}{8\pi} \frac{(k-m)!}{(k+m)!}} P_k^m (x) \qquad (1)$$

where P_k^m (x) are the standardly defined Associated Legendre Polynomials. $\bar{P}_k^m$ (x) are the Normalized Associated Legendre Polynomials so that K_k^m satisfy:

$$\int_1^1 K_k^m(x)\, K_{k'}^m(x)\, dx = \frac{1}{4\pi} \delta_{kk'} \qquad (2)$$

For all $0 \leq m \leq k$ and all $0 \leq k \leq 12$, the functions $|K_k^m (x)| \leq |$ and are hence in scale.

The routine operates in three parts:

(a) The unnormalized Legendre Polynomials $P_k \equiv P_k^0$ are first computed using:

$$1/2\ P_0(x) = 1/2$$

$$1/2\ P_1(x) = 1/2\ x$$

$$1/2\ P_k(x) = x[1/2\ P_{k-1}(x)] + \frac{k-1}{k} \left[x[1/2\ P_{k-1}(x)] - 1/2\ P_{k-2}(x)\right] \qquad (3)$$

and inserting 1/2 P_k(x) in the proper place in the table at S3.

(b) Each 1/2 $P_k(x)$ is multiplied by $2\sqrt{\frac{2k+1}{8\pi}}$ to form

$$K_k^0(x) = 2\sqrt{\frac{2k+1}{8\pi}}\ [1/2\ P_k(x)] \qquad (4)$$

(c) The recursion formula

$$\sqrt{(1-x^2)}\ K_k^m(x) = \sqrt{\frac{(2k+1)(k+m-1)}{(2k-1)(k+m)}}\ K_{k-1}^{m-1}(x) - \sqrt{\frac{k-m+1}{k+m}}\ x\ K_k^{m-1}(x) \qquad (5)$$

is then used to compute the $K_k^m(x)$ for M = 0 and fill in the table as S3.

DISCUSSION OF ERROR

This program suffers from the very great defect common to all programs which generate a sequence of functions using recursion relations, namely the accumulation and magnification of error at each step. From equations (3) and (5) it can be seen that the error in K_k^0 increases with k and that the error in K_k^m for fixed k increases with m. The latter increase is much more pronounced than the former. In addition the error will be much worse for $|x| \sim 1$ due to $\sqrt{1-x^2}$ in equation (5). In fact for $x < -.985$ or $x > .99$, the K_k^m lose so much significance for k and m large that division hang-ups will occur. (if $|x| < 2^{-19}$ a division hang-up will occur in the attempt to compute $\sqrt{1-x^2}$). These possibilities dictate that this routine be used with a great measure of care, skill, and caution. Representative K_k^m calculated by this routine for representative values of x have been checked against the 10 place tables of Associated Legendre Function of Zaki Mursi, Fouad I University, 1941. These errors are given in Table I in units of 10^{-10}. Note the extremely poor results for x near 1 and for k, m large. One saving feature in this respect is that K_k^m (for $m \neq 0$) $\rightarrow 0$ as $|x| \rightarrow 1$. Hence in the range near 1 in which this program is inapplicable, it may be permissible in some problems to set $K_k^m = 0$ for x in this troublesome range. For a fixed k this becomes a better approximation as m increases which arejust the cases (i.e. m increasing for fixed k) in which this routine becomes increasingly bad.

PROGRAMMED BY	J. N. Snyder
APPROVED BY	JPNash
DATE	November 18, 1955

JNS/mge
11/18/55

TABLE I: ERROR IN THE FUNCTIONS K_k^m (in units of 10^{-10})

FUNCTION	X = 0	X = .3	X = .6	X = .9	X = .95	X = .98	X = .985
K_1^1	< 1	< 1	3	9	1	7	2
K_2^1		3	1	10	9	5	8
K_2^2	< 1	< 1	1	.3	6	2	2
K_3^1	< 1	< 1	< 1	6	< 1	1	7
K_3^2		< 1	< 1	< 1	.1	.4	2
K_3^3	< 1	< 1	< 1	.7	2	9	11
K_4^1		1	< 1	< 1	4	4	1
K_4^2	< 1	.3	< 1	<< 1	2	10	1
K_4^4	< 1	< 1	< 1	2	9	50	62
K_5^1	< 1	4	< 1	2	6	5	2
K_5^3	2	1	1	1	5	5	5
K_5^5	8	6	3	5	40	300	347
K_6^1		1	1	4	1	4	1
K_6^3		2	1	5	3	5	7
K_6^6	7	5	2	12	130	1610	1720
K_7^1	3	< 1	2	5	9	2	4
K_7^4		1	3	6	9	10	51
K_7^7	24	17	5	24	450	8400	12400
K_8^1		4	< 1	1	1	1	4
K_8^4		5	1	2	5	27	42

(Table I continued on next page)

FUNCTION	X = 0	X = .3	X = .6	X = .9	X = .95	X = .98	X = .985
K_8^8	10	8	25	50	1500	37000	16200
K_9^1	< 1	2	1	4	1	3	2
K_9^5	16	8	7	5	24	133	620
K_9^9	87	57	50	105	4200	42000	300,000
K_{10}^1		1	1	2	2	1	4
K_{10}^5		50	50	230	76	44	1100
K_{10}^{10}	2	120	177	270	10,100	1,140,000	worthless

LOCATION	ORDER		NOTES PAGE 1
0	40 3F		Store argument in 3
	41 5F		clear k_n storage location
1	41 F		Create a zero
	K5 F		Plant link
2	42 67L		
	10 20F		Plant k_n
3	42 SF		
	49 S3		Set $1/2\ P_0 = 1/2$ in S3
4	L5 3F		
	10 1F		Set $1/2\ P_1 = 1/2\ x$ in 1S3
5	40 1S3		
	F5 F		Set $k = 1$
6	40 4F		
	L5 3L		Set initial
7	42 13L		addresses to S3
	42 28L		
8	42 62L		
	L5 13L	Form 20	Set address of P_{k-2}
9	42 17L		
	L4 4F		Set address of P_{k-1}
10	42 13L		
	F4 4F		Set address of P_k
11	42 19L		
	F5 4F		Advance $k \rightarrow k + 1$
12	40 4F		
	F0 5F		$k - k_n$
13	32 20L		⊕ $k \geq k_n$, Done with P's
	50 ( )F	By 7 and 10	⊖ Not done with P's
14	7J 3F		$x\ 1/2\ P_{k-1} \rightarrow 1$
	40 1F		
15	L5 4F		
	F0 F		$\frac{k-1}{k}$ in Q
16	50 F		
	66 4F		

LOCATION	ORDER		NOTES PAGE 2
17	L5 1F		
	L0 ( )F	By 9	complete 1/2 P_k
18	40 2F		
	7J 2F		
19	L4 1F		
	40 ( )F	By 11	Store 1/2 P_k
20	22 8L		Repeat cycle
	41 4F	From 13	Set k counter to 0
21	L5 28L	From 32	
	L4 4F		Advance
22	42 28L		addresses
	42 29L		
23	41 F		Clear 0
	50 F		Clear Q
24	L5 4F		
	F4 4F		$\sqrt{\frac{(2k+1)}{8\pi}}$ in A
25	00 34F		
	66 70L		
26	S5 F		
	50 26L		$\sqrt{\frac{(2k+1)}{8\pi}}$ in 2
27	26 71L		
	19 1F		2^{-2} in A
28	50 2F		
	74 ( )F	By 7 and 22	$\sqrt{\frac{2k+1}{8\pi}}$ in Q
29	00 1F		
	40 ( )F	By 22	Store K_k
30	F5 4F		
	40 4F		advance k counter
31	F0 5F		$k - k_n$
	32 32L		⊕ Done normalizing
32	26 21L		⊖ Continues
	50 3F	From 31	

LOCATION	ORDER		NOTES PAGE 3
33	71 3F		$-x^2$ in A
	L4 691		$1 - x^2$ in A
34	40 1F		NSP $(1 - x^2) \rightarrow 1$
	S5 F		
35	40 F		LSP $(1 - x^2) \rightarrow 0$
	50 35L		
36	22 71L		
	40 8F		$\sqrt{1 - x^2}$ in 8
37	41 F		Clear 0
	F5 F		
38	40 4F		Set k = 1
	40 7F	From 68	Set m = 1
39	F5 62L		
	42 42L	From 65	Plant address of K_k^{m-1}
40	F4 F		Plant address of K_k^m
	42 62L		
41	F0 4F		Plant address of K_{k-1}^{m-1}
	42 57L		
42	50 3F		
	7J (′)F	By 39	xK_k^{m-1} to 6
43	40 6F		
	L5 4F		
44	L4 7F		$(k + m) \rightarrow 9$
	40 9F		
45	F0 F		$(k + m - 1) \rightarrow 1$
	40 1F		
46	L4 4F		
	L0 7F		$(2k - 1) \rightarrow Q$
47	10 39F		
	75 9F		$(k + m)(2k - 1) \rightarrow 2$
48	S5 F		
	40 2F		

LOCATION	ORDER		NOTES PAGE 4
49	F5 4F		
	L4 4F		$(2k + 1) \to Q$
50	10 39F		
	75 1F		$1/4\ (k + m - 1)\ (2k + 1)$ inA
51	00 37F		
	66 2F		
52	S5 F		
	50 52L		$1/2 \sqrt{\dfrac{(k+m-1)(2k+1)}{(1+m)\ (2k\ -1)}} \to 10$
53	26 71L		
	40 10F		
54	F5 4F		
	L0 7F		$(k - m + 1)$ in A
55	50 F		Clear Q
	66 9F		
56	S5 F		
	50 56L		$\sqrt{\dfrac{k - m + 1}{k + m}}$ in 2
57	26 71L		
	50 ( )F	By 41	K_{k-1}^{m-1} in Q
58	75 10F		
	00 1F		$\sqrt{\dfrac{(k+m-1)(2k+1)}{(k+m)(2k-1)}}$ in A
59	40 10F		NSP ["] in 10
	S5 F		LSP ["] in A
60	50 2F		
	70 6F		
61	L4 10F		
	66 8F		
62	S5 F		
	40 ( )F	By 8 and 40	Store K_k^m
63	F5 7F		
	40 7F		Advance $m \to m + 1$
64	F0 4F		$m - k$ in A
	36 66L		⊕ $m \geq k$, done this k

LOCATION	ORDER		NOTES PAGE 5
65	L5 62L		⊖ Not done this k
	22 39L		Repeat
66	F5 4F	From 64	
	40 4F		advance k → k + 1
67	F0 5F		k - k_n in A
	32 ( )F	By 2	⊕ Done, enter Link
68	F5 F		⊖ not done,
	22 38L		Set m - 1 and repeat
69	80 F		
	00 F	-	- 1 = 1
70	40 F		
	00 285 398 163 397J		$\frac{8\pi}{32} = \frac{\pi}{4}$
71	50 72L		
	26 999F		Interlude
72	00 F		
	00 71L		to load
73	00 F		Routine R - 1 116
	26 71L		at 74L
	26 1N		
Routine R1 - 116 Square Root			